ON THE RELATIONSHIP BETWEEN $\mu$-ININVARIANT MEASURES AND QUASI-STATIONARY DISTRIBUTIONS FOR CONTINUOUS-TIME MARKOV CHAINS

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Abstract

In a recent paper, van Doorn (1991) explained how quasi-stationary distributions for an absorbing birth–death process could be determined from the transition rates of the process, thus generalizing earlier work of Cavender (1978). In this paper we shall show that many of van Doorn's results can be extended to deal with an arbitrary continuous-time Markov chain over a countable state space, consisting of an irreducible class, $C$, and an absorbing state, 0, which is accessible from $C$. Some of our results are extensions of theorems proved for honest chains in Pollett and Vere-Jones (1992).

In Section 3 we prove that a probability distribution on $C$ is a quasi-stationary distribution if and only if it is a $\mu$-invariant measure for the transition function, $P$. We shall also show that if $m$ is a quasi-stationary distribution for $P$, then a necessary and sufficient condition for $m$ to be $\mu$-invariant for $Q$ is that $P$ satisfies the Kolmogorov forward equations over $C$. When the remaining forward equations hold, the quasi-stationary distribution must satisfy a set of 'residual equations' involving the transition rates into the absorbing state. The residual equations allow us to determine the value of $\mu$ for which the quasi-stationary distribution is $\mu$-invariant for $P$. We also prove some more general results giving bounds on the values of $\mu$ for which a convergent measure can be a $\mu$-subinvariant and then $\mu$-invariant measure for $P$. The remainder of the paper is devoted to the question of when a convergent $\mu$-subinvariant measure, $m$, for $Q$ is a quasi-stationary distribution. Section 4 establishes a necessary and sufficient condition for $m$ to be a quasi-stationary distribution for the minimal chain. In Section 5 we consider 'single-exit' chains. We derive a necessary and sufficient condition for there to exist a process for which $m$ is a quasi-stationary distribution. Under this condition all such processes can be specified explicitly through their resolvents. The results proved here allow us to conclude that the bounds for $\mu$ obtained in Section 3 are, in fact, tight. Finally, in Section 6, we illustrate our results by way of two examples: regular birth–death processes and a pure-birth process with absorption.

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A resolvent that satisfies (2.8) is called a \( Q \)-resolvent. The resolvent, \( \Phi(\cdot) = (\phi_q(\cdot), 0, 1, \cdots) \). We shall suppose that one is given a stable, conservative \( q \)-matrix over \( \mathcal{S} \), that is a set, \( \mathcal{Q} = (q_{ij}, i, j \in \mathcal{S}) \), of real numbers which satisfy \( 0 \leq q_{ij} < \infty, j \neq i \), and \( \sum_{j \neq i} q_{ij} = -q_{ii} < \infty \); as is the convention, we shall set \( q_i = -q_{ii} \) for each \( i \in \mathcal{S} \).

A set of real-valued functions, \( P(\cdot) = (p_{ij}(\cdot), i, j \in \mathcal{S}) \), defined on \([0, \infty)\) is called a standard transition function if

\[
\begin{align*}
(1.1) & \quad \quad p_{ij}(t) \geq 0, \quad i, j \in \mathcal{S}, \quad t \geq 0, \\
(1.2) & \quad \quad \sum_{j \in \mathcal{S}} p_{ij}(t) \leq 1, \quad i \in \mathcal{S}, \quad t \geq 0, \\
(1.3) & \quad \quad p_{ij}(s + t) = \sum_{k \in \mathcal{S}} p_{ik}(s) p_{kj}(t), \quad i, j \in \mathcal{S}, \quad s, t \geq 0, \\
(1.4) & \quad \quad p_{ij}(0) = \delta_{ij} = \lim_{t \downarrow 0} p_{ij}(t), \quad i, j \in \mathcal{S}.
\end{align*}
\]

\( P \) is then honest if equality holds in (1.2) for some (and then for all) \( t > 0 \). It is called a \( Q \)-function if \( p_{ij}(0+) = q_{ij} \) for each \( i, j \in \mathcal{S} \).

Under the conditions we have imposed, each \( Q \)-function, \( P \), satisfies the backward differential equations

\[
(BE_{ij}) \quad \quad \quad \quad p'_{ij}(t) = \sum_{k \in \mathcal{S}} q_{ik} p_{kj}(t), \quad t > 0,
\]

for all \( i, j \in \mathcal{S} \), but might not satisfy the forward differential equations

\[
(FE_{ij}) \quad \quad \quad \quad p'_{ij}(t) = \sum_{k \in \mathcal{S}} p_{ik}(t) q_{kj}, \quad t > 0,
\]

for all \( i, j \in \mathcal{S} \). One can assert only that \( P \) satisfies the forward differential inequalities

\[
(FI_{ij}) \quad \quad \quad \quad \quad \quad p_{ij}(t) \geq \sum_{k \in \mathcal{S}} p_{ik}(t) q_{kj}, \quad t > 0;
\]

these hold for all \( i, j \in \mathcal{S} \), even if \( Q \) is not conservative (see Reuter (1957)). Later we shall show that if \( m \) is a quasi-stationary distribution on \( \mathcal{C} \), where \( \mathcal{C} \) is a transient class, then for \( m \) to be a \( \mu \)-invariant measure for \( Q \), it is necessary that \( P \) satisfies (\( FE_{ij} \)) over \( \mathcal{C} \).

2. Invariant measures

If \( \mathcal{C} \) is a subset of \( \mathcal{S} \) and \( \mu \) is some fixed non-negative real number, then a collection of positive numbers, \( m = (m_j, j \in \mathcal{C}) \), is called a \( \mu \)-subinvariant measure on \( \mathcal{C} \) for \( Q \) if

\[
(2.1) \quad \quad \quad \quad \quad \quad \sum_{i \in \mathcal{C}} m_i q_{ij} \leq -\mu m_j, \quad j \in \mathcal{C},
\]

and \( \mu \)-invariant if equality holds for all \( j \in \mathcal{C} \). In contrast, \( m \) is said to be \( \mu \)-subinvariant on \( \mathcal{C} \) for \( P \), where \( P \) is a standard transition function, if

\[
(2.2) \quad \quad \quad \quad \quad \quad \sum_{i \in \mathcal{C}} m_i p_{ij}(t) \leq \exp(-\mu t) m_j, \quad j \in \mathcal{C}, \quad t \geq 0,
\]
1. Introduction

We shall be concerned with continuous-time Markov chains that take values in a countable state space $S$, which, for convenience, we shall enumerate as $S = \{1, 2, \cdots\}$ and $\mu$-invariant if equality holds for all $j \in C$ and $t \geq 0$. For simplicity, we shall say that a given measure, $m$, on $C$ is convergent if $\sum_{i \in C} m_i < \infty$.

The relationship between (2.1) and (2.2) has been resolved completely for the minimal $Q$-function, that is the minimal solution to $(BE_{ij})$, $i, j \in S$, satisfying $p_{ij}(0+) = q_{ij}$, $i, j \in S$; see Tweedie (1974) and Pollett (1986), (1988). In Section 3 we shall extend these results in order to deal with $Q$-functions other than $F$.

As we shall be concerned with absorbing Markov chains, it will be necessary to make some further assumptions about $Q$. Firstly, we shall assume that $0$ is the sole absorbing state. Thus, if $P$ is any $Q$-function we shall require $p_{0j}(t) = \delta_{0j}$; this is equivalent to assuming that $q_{0j} = 0$. Secondly, and solely to simplify the exposition, we shall assume that $C = \{1, 2, \cdots\}$ is irreducible for the minimal $Q$-function, and hence for any $Q$-function. Finally, we shall assume that $q_{i0} > 0$ for at least one $i \in C$. This guarantees that there is a positive probability of absorption, that is, $p_{i0}(t) > 0$ for all $t > 0$; it will not be necessary to assume that absorption is certain.

In some instances it is convenient for us to use Laplace transforms. If $P$ is a $Q$-function, then its resolvent, $\Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S)$, is given by

\begin{equation}
\psi_{ij}(\alpha) = \int_0^\infty \exp(-\alpha t)p_{ij}(t)\, dt, \quad i, j \in S.
\end{equation}

If $i, j \in C$, this integral converges for all $\alpha > -\lambda_P(C)$, where $\lambda_P(C)$ is the decay parameter of $C$ for $P$ (see Kingman (1963)). In particular, since $C$ is irreducible, the Laplace transforms of $p_{ij}$, for $i, j \in C$, have the same abscissa of convergence. Notice also that, since 0 is an absorbing state, $\psi_{0j}(\alpha) = \delta_{0j}/\alpha$. Analogous to (1.1)–(1.4), $\Psi$ satisfies

\begin{equation}
\psi_{ij}(\alpha) \geq 0, \quad i, j \in S, \quad \alpha > 0,
\end{equation}

\begin{equation}
\sum_{j \in S} \alpha \psi_{ij}(\alpha) \leq 1, \quad i \in S, \quad \alpha > 0,
\end{equation}

the ‘resolvent equation’

\begin{equation}
\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in S} \psi_{ik}(\alpha)\psi_{kj}(\beta) = 0, \quad i, j \in S, \quad \alpha, \beta > 0,
\end{equation}

and

\begin{equation}
\lim_{\alpha \to \infty} \alpha \psi_{ij}(\alpha) = \delta_{ij}, \quad i, j \in S.
\end{equation}

Indeed, any $\Psi$ which satisfies (2.4)–(2.7) is the resolvent of a standard transition function (see Reuter (1959) and (1962)). Further, (2.5) is satisfied with equality if and only if $P$ is honest, in which case the resolvent is said to be honest. The $q$-matrix of $P$ can be recovered from $\Psi$ using the following identity:

\begin{equation}
q_{ij} = \lim_{\alpha \to \infty} \alpha(\alpha \psi_{ij}(\alpha) - \delta_{ij}).
\end{equation}

A resolvent that satisfies (2.8) is called a $Q$-resolvent. The resolvent, $\Phi(\cdot) = (\phi_{ij}(\cdot)$,
On the relationship between \( \mu \)-invariant measures and quasi-stationary distributions

4.1

\( i, j \in S \), of the minimal \( Q \)-function, has itself a minimal interpretation (see Reuter (1957) and (1959)); for this reason \( \Phi \) is called the minimal \( Q \)-resolvent.

We shall now explain how \( \mu \)-invariant and \( \mu \)-subinvariant measures can be identified using resolvents. If \( P \) is a \( Q \)-function with resolvent \( \Psi \) and \( m = (m_j, j \in C) \) is a \( \mu \)-subinvariant measure on \( C \) for \( P \), where of necessity \( \mu \leq \lambda_p(C) \) (see Lemma 4.1 of Vere-Jones (1967)), then, since the integral in (2.3) converges for all \( \alpha > -\lambda_p(C) \), we have that, for all \( j \in C \) and \( \alpha > 0 \),

\[
(2.9) \quad \sum_{i \in S} m_i \alpha \psi_i(\alpha - \mu) \leq m_j,
\]

with equality for all \( j \) and \( \alpha \) if \( m \) is \( \mu \)-invariant on \( C \) for \( P \); we shall refer to \( m \) as being \( \mu \)-subinvariant on \( C \) for \( \Psi \) if (2.9) is satisfied and \( \mu \)-invariant on \( C \) for \( \Psi \) if it is satisfied with equality. The following result establishes a characterization of \( \mu \)-invariance and \( \mu \)-subinvariance on \( C \) for \( P \) in terms of \( \Psi \). It is a simple extension of Lemma 4.1 of Pollett (1991b), which deals with the \( \mu = 0 \) case, and, since the proof is similar, it will be omitted.

**Lemma 2.1.** Let \( m \) be a measure on \( C \) and suppose that \( P \) has resolvent \( \Psi \). Then, if \( m \) is \( \mu \)-subinvariant on \( C \) for \( P \), it is \( \mu \)-subinvariant on \( C \) for \( \Psi \) and \( \mu \)-invariant on \( C \) for \( \Psi \) if it is \( \mu \)-invariant on \( C \) for \( P \). Conversely, if \( \mu \leq \lambda_p(C) \) and \( m \) is \( \mu \)-subinvariant on \( C \) for \( \Psi \), then \( m \) is \( \mu \)-subinvariant on \( C \) for \( P \) and \( \mu \)-invariant on \( C \) for \( P \) if it is \( \mu \)-invariant on \( C \) for \( \Psi \).

3. Quasi-stationary distributions

We begin by defining the notion of a quasi-stationary distribution and then study the relationship between quasi-stationary distributions and \( \mu \)-invariant and \( \mu \)-subinvariant measures. The definition we shall use is the one introduced by van Doorn (1991).

**Definition 3.1.** Let \( m = (m_j, j \in C) \) be a proper probability distribution over \( C \) and define \( p(\cdot) = (p_i(\cdot), j \in S) \) by

\[
(3.1) \quad p_j(t) = \sum_{i \in C} m_i p_i(t), \quad j \in S, \quad t \geq 0.
\]

Then \( m \) is a **quasi-stationary distribution on \( C \) for \( P \)** if, for all \( t > 0 \),

\[
(3.2) \quad \frac{p_j(t)}{\sum_{i \in C} p_i(t)} = m_j, \quad j \in C.
\]

**Remark.** If \( (X(t), t \geq 0) \) is a continuous-time Markov chain with transition function \( P \) and with initial distribution \( m \), then (3.2) stipulates that the state probabilities at time \( t \), conditional on the chain being in \( C \) at \( t \), do not vary with \( t \).
The relationship between quasi-stationary distributions and $\mu$-invariant measures for $P$ is exceedingly simple, as the following result demonstrates. Its conclusion is implicit in the proof of Theorem 2 of Vere-Jones (1969). However, note that here the premise that $P$ be honest is not needed.

**Proposition 3.1.** Let $P$ be an arbitrary $Q$-function and suppose that $m$ is a proper probability distribution on $C$. Then $m$ is a quasi-stationary distribution on $C$ for $P$ if and only if, for some $\mu > 0$, $m$ is $\mu$-invariant on $C$ for $P$.

**Proof.** Suppose that, for some $\mu > 0$, $m$ is $\mu$-invariant on $C$ for $P$ and define $p(\cdot) = (p_j(\cdot), j \in C)$ by (3.1). Then, $p_j(t) = \exp(-\mu t)m_j$, and so, since $m$ is a proper distribution, (3.2) holds. Conversely, suppose that $m$ is a quasi-stationary distribution on $C$ for $P$. Then $p_j(t) = g(t)m_j$, where $g(t) = \sum_{j \in C} p_j(t)$, or, equivalently,

$$\sum_{j \in C} m_i p_{ij}(t) = g(t)m_j, \quad j \in C, \quad t > 0.$$ 

We show that $g(t) = \exp(-\mu t)$ for some $\mu > 0$. To do this, we use (1.3). On multiplying by $m_i$, summing over $i \in C$ and noting that 0 is an absorbing state, we obtain

$$p_j(s + t) = \sum_{k \in C} p_k(s)p_{kj}(t), \quad j \in C, \quad s, t \geq 0.$$ 

After substituting the expression for $p(\cdot)$ in this equation, we see that $g$ satisfies the familiar functional equation: $g(s + t) = g(s)g(t)$, $s, t \geq 0$. Note that we must have $0 < g(t) \leq 1$, since $p_j(t) \geq m_j p_{ij}(t) > 0$ and

$$g(t) = \sum_{j \in C} \sum_{i \in C} m_i p_{ij}(t) = \sum_{i \in C} m_i \sum_{j \in C} p_{ij}(t) \leq 1.$$ 

It follows that $g(t) = \exp(-\mu t)$, for some $\mu \geq 0$. Finally, since for at least one $i \in C$, $p_{0i}(t) > 0$ for all $t > 0$, we have that $g(t) < 1$ for all $t > 0$, and so the case $\mu = 0$ is excluded.

We can now identify the relationship between quasi-stationary distributions and $\mu$-invariant measures on $C$ for $Q$. We need the following precursory result; since it is a simple extension of Theorem 1 of Pollett and Vere-Jones (1992), we omit the proof.

**Theorem 3.1.** If $m$ is a $\mu$-subinvariant measure on $C$ for $P$ then $m$ is a $\mu$-subinvariant measure on $C$ for $Q$. A necessary condition for $m$ to be a $\mu$-invariant measure on $C$ for $Q$ is that $P$ satisfies $(FE_{ij})$ over $C$. If $m$ is $\mu$-invariant on $C$ for $P$, then this condition is also sufficient.

Proposition 3.1 and Theorem 3.1 combine to give the following simple result.

**Corollary 3.1.** Suppose that $m$ is a quasi-stationary distribution on $C$ for $P$. Then for some $\mu > 0$, $m$ is a $\mu$-subinvariant measure on $C$ for $Q$ and $\mu$-invariant if and only if $P$ satisfies $(FE_{ij})$ over $C$.  

When \( m \) is a quasi-stationary distribution on \( C \) for \( P \), we can often be precise about the value of \( \mu \) for which \( m \) is a \( \mu \)-invariant measure. For example, we shall see that if \( P \) is honest and satisfies the forward equations (\( FE_\mu \)) over \( S \), then \( \mu \) must take the value \( \sum_{i \in C} m_i q_{i0} \), a quantity which represents the conditional probability flux into the absorbing state for a Markov chain with transition function \( P \), that is conditional on the chain being in \( C \) (see, for example, Section 7.1 of Kelly and Pollett (1983)). In the general case we cannot be so precise, but we can specify bounds on the range of values of \( \mu \). These are expressed in terms of the aforementioned probability flux and the absorption probabilities, \( a_i^P, i \in C \), given by \( a_i^P = \lim_{t \to \infty} p_{i0}(t) \).

In preparation we need the following lemma.

**Lemma 3.1.** Let \( P \) be a \( Q \)-function and suppose that, for some \( \mu > 0 \), there exists a convergent \( \mu \)-subinvariant measure on \( C \) for \( P \). Then \( a_i^P = h_i^P \), for all \( i \in C \), where

\[
h_i^P = \lim_{t \to \infty} \sum_{j \in S} p_{ij}(t), \quad i \in C.
\]

**Proof.** Since \( m \) is a \( \mu \)-subinvariant measure on \( C \) for \( P \), we have, in particular, that \( m_j p_{ij}(t) \leq \exp (-\mu t) m_i \), for \( i, j \in C \). It follows that

\[
\sum_{j \in C} p_{ij}(t) \leq \exp (-\mu t) \frac{1}{m_i} \sum_{j \in C} m_j, \quad i \in C,
\]

and hence, because \( m \) is convergent and \( \mu > 0 \),

\[
\lim_{t \to \infty} \sum_{j \in C} p_{ij}(t) = 0,
\]

for all \( i \in C \). The result then follows immediately.

**Remark.** Since, for each \( i \in S \), \((p_{ij}(-), j \in S)\) is a bounded and normalized \( P \)-entrance law, \( h_i^P \) is the probability that the lifetime of a Markov chain with transition function \( P \) is infinite, given that it starts in state \( i \) (see Proposition 1 of Lamb (1971)). Thus Lemma 3.1 shows that, if a convergent \( \mu \)-subinvariant measure exists, the chain must ultimately leave \( C \) starting in \( i \in C \).

**Theorem 3.2.** Let \( P \) be a \( Q \)-function and suppose that, for some \( \mu > 0 \), \( m \) is a convergent \( \mu \)-invariant measure on \( C \) for \( P \). Then

\[
\mu \geq \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i a_i^P}.
\]

**Proof.** We shall use the forward differential inequalities. On integrating (\( FI_{i0} \)) we obtain

\[
p_{i0}(t) \geq \sum_{k \in C} \int_0^t p_{ik}(s) q_{k0} ds, \quad i \in C.
\]
On multiplying by $m_i$ and summing this inequality over $i \in C$ we see that, since $m$ is a convergent $\mu$-invariant measure on $C$ for $P$, it is necessary that $\sum_{k \in C} m_k q_{k0}$ converges. Moreover, we have that, for all $t \geq 0$,

$$\sum_{i \in C} m_i p_{i0}(t) \geq \frac{1}{\mu} (1 - \exp(-\mu t)) \sum_{k \in C} m_k q_{k0}. \tag{3.4}$$

Now, it is clear that

$$\sum_{i \in C} m_i \sum_{j \in C} p_{ij}(t) = \exp(-\mu t) \sum_{i \in C} m_i, \quad t \geq 0,$$

and so, if $e^P(\cdot) = (e^P_i(\cdot), i \in S)$ is the dishonesty function, defined by

$$e^P_i(t) = 1 - \sum_{j \in S} p_{ij}(t), \quad t \geq 0,$$

then we must also have that

$$\sum_{i \in C} m_i p_{i0}(t) = (1 - \exp(-\mu t)) \sum_{i \in C} m_i - \sum_{i \in C} m_i e^P_i(t), \quad t \geq 0. \tag{3.5}$$

On combining (3.4) and (3.5) we get

$$\frac{1}{\mu} (1 - \exp(-\mu t)) \left( \mu \sum_{i \in C} m_i - \sum_{i \in C} m_i q_{i0} \right) \geq \sum_{i \in C} m_i e^P_i(t)$$

and, on letting $t \to \infty$ and using bounded convergence, we obtain

$$\mu \sum_{i \in C} m_i \geq \sum_{i \in C} m_i q_{i0} + \mu \sum_{i \in C} m_i e^P_i,$$

where $e^P_i = \lim_{t \to \infty} e^P_i(t)$. But $e^P_i = 1 - h^P_i$ and, by Lemma 3.1, $h^P_i = a^P_i$ for all $i \in C$. Thus

$$\mu \sum_{i \in C} m_i a^P_i \geq \sum_{i \in C} m_i q_{i0}. \tag{3.6}$$

Since $q_{i0} > 0$ for at least one $i \in C$, the right-hand side of this last inequality must be strictly positive, and so (3.3) follows immediately.

**Remark.** If the Markov chain with transition function $P$ has initial distribution $m$, then $\sum_{i \in C} m_i a^P_i$ is the probability that the chain eventually reaches the absorbing state, or, equivalently, that its lifetime is infinite. Note that when $P$ is dishonest one cannot rule out the possibility that $a^P_i = 0$ for all $i \in C$, but that under the conditions of Theorem 3.2, $a^P_i > 0$ for at least one value of $i$.

Equation (3.5) is the appropriate generalization, to the case when $P$ might not be honest, of the 'residual equations' for $P$ identified in Pollett and Vere-Jones (1992). The analogous residual equations for $Q$ do not even hold when $P$ is honest. One requires a further condition. To see this, we shall first establish a residual inequality, which is the reverse of (3.6).
Theorem 3.3. Let $P$ be a $Q$-function which satisfies (FE$_{i0}$) over $C$, and suppose that, for some $\mu > 0$, $m$ is a convergent $\mu$-subinvariant measure on $C$ for $P$. Then

$$\mu \sum_{i \in C} m_i a_i^p \leq \sum_{i \in C} m_i q_{i0},$$

and, if $P$ is honest

$$\mu \leq \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i}.$$

Proof. Since $P$ satisfies (FE$_{i0}$), $i \in C$, we have that

$$p_{i0}(t) = \sum_{k \in C} \int_0^t p_{ik}(s) q_{k0} \, ds, \quad i \in C.$$ 

On multiplying by $m_i$, summing over $i \in C$ and using the fact that $m$ is a $\mu$-subinvariant measure on $C$ for $P$, we get

$$\sum_{i \in C} m_i p_{i0}(t) \leq \frac{1}{\mu} (1 - \exp(-\mu t)) \sum_{k \in C} m_k q_{k0},$$

for all $t \geq 0$. Since now

$$\sum_{i \in C} m_i \sum_{j \in C} p_{ij}(t) \leq \exp(-\mu t) \sum_{i \in C} m_i, \quad t \geq 0,$$

we also have

$$\sum_{i \in C} m_i p_{i0}(t) \geq (1 - \exp(-\mu t)) \sum_{i \in C} m_i - \sum_{i \in C} m_i e_i^p(t), \quad t \geq 0.$$ 

Thus

$$\frac{1}{\mu} (1 - \exp(-\mu t)) \left( \sum_{i \in C} m_i - \sum_{i \in C} m_i q_{i0} \right) \leq \sum_{i \in C} m_i e_i^p(t)$$

and so, on letting $t \to \infty$, we obtain

$$\mu \sum_{i \in C} m_i \leq \sum_{i \in C} m_i q_{i0} + \mu \sum_{i \in C} m_i e_i^p,$$

equivalently,

$$\mu \sum_{i \in C} m_i h_i^p \leq \sum_{i \in C} m_i q_{i0}.$$ 

Using Lemma 3.1 we arrive at (3.7). The last part follows because, if $P$ is honest, $h_i^p = 1$ for all $i \in C$.

Combining the results of Theorem 3.2 and Theorem 3.3, we obtain the following generalization of Theorem 2 of Pollett and Vere-Jones (1992).
Corollary 3.2. Let $P$ be a $Q$-function which satisfies (FE$_{i0}$) over $C$ and suppose that, for some $\mu > 0$, $m$ is a convergent $\mu$-invariant measure on $C$ for $P$. Then

$$\mu = \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i a_i^P}.$$ 

Moreover,

$$\mu = \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i}$$

if and only if $P$ is honest.

The results of the two corollaries of this section lead us to the following theorem.

Theorem 3.4. Suppose that $m$ is a quasi-stationary distribution on $C$ for $P$ and that $P$ satisfies (FE$_{i0}$) over $S$. Then $m$ is a $\mu$-invariant measure on $C$ for $Q$, where

$$\mu = \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i a_i^P},$$

and

$$\mu = \sum_{i \in C} m_i q_{i0}$$

if and only if $P$ is honest.

4. Quasi-stationary distributions for the minimal process

From a practical point of view, one usually begins with a convergent $\mu$-invariant measure, $m$, on $C$ for $Q$ and then seeks to determine whether or not it is a quasi-stationary distribution for the minimal $Q$-function, $F$. Since $F$ satisfies the forward equations, Theorem 3.4 tells us that for $m$ to be a quasi-stationary distribution it is necessary that $m$ be a $\mu$-invariant probability distribution on $C$ for $Q$ and that

$$\mu = \frac{\sum_{i \in C} m_i q_{i0}}{\sum_{i \in C} m_i a_i^F},$$

where $a_t^F = \lim_{t \to \infty} f_{i0}(t)$. The following result deals with the converse.
Theorem 4.1. Let $F(\cdot) = (f_{ij}(\cdot), i, j \in S)$ be the minimal $Q$-function and suppose that, for some $\mu > 0$, $m$ is a convergent $\mu$-subinvariant measure on $C$ for $Q$. Then, $m$ is a $\mu$-invariant measure on $C$ for $F$ if and only if

\begin{equation}
\mu \sum_{i \in C} m_i a_i^f = \sum_{i \in C} m_i q_{i0}.
\end{equation}

When this condition holds, $m$ is $\mu$-invariant on $C$ for $Q$.

Proof. Since $F$ satisfies $(FE_{ij})$ over $C$, Corollary 3.2 establishes the necessity of (4.1). So, suppose that (4.1) holds. We shall show that $m$ is a $\mu$-invariant measure on $C$ for $F$. Let $Q^* = (q^*_i, i, j \in C)$ be the $\mu$-reverse of $Q$ with respect to $m$ and let $F^*(\cdot) = (f^*_i(\cdot), i, j \in S)$ be the minimal $Q^*$-function. Then, by Lemma 3.3 of Pollett (1988), we have that $m_i f_i(t) = \exp \left( -\mu t \right) m_i q_{i0}$, for $i, j \in C$. Summing over $i$ shows that $m$ is $\mu$-subinvariant for $F$ and $\mu$-invariant if and only if $F^*$ is honest. Thus, the proof is complete if we can show that $F^*$ is honest.

We proceed using Laplace transforms. Let $Q(\cdot) = (q_i(\cdot), i, j \in S)$ be the resolvent of $F$ and let $Q^*(\cdot) = (q_i^*(\cdot), i, j \in C)$ be the resolvent of $F^*$. Then, we have

\begin{equation}
m_i \phi_i(\alpha) = m_i \phi_i^*(\alpha + \mu), \quad i, j \in C,
\end{equation}

and, summing over $j \in C$ gives

\begin{equation}
m_i \alpha \phi_{i0}(\alpha) = m_i (1 - z_i(\alpha)) - \sum_{j \in C} m_j \alpha \phi_j^*(\alpha + \mu), \quad i \in C,
\end{equation}

where $z(\cdot) = (z_i(\cdot), i \in C)$ is given by $z_i(\alpha) = 1 - \sum_{j \in C} \alpha \phi_j^*(\alpha)$. Thus, on summing (4.3) over $i$ and using the fact that $m$ is convergent, we deduce that

\begin{equation}
\sum_{i \in C} m_i \alpha \phi_{i0}(\alpha) + \sum_{i \in C} m_i z_i(\alpha) = \frac{\mu}{\alpha + \mu} \sum_{i \in C} m_i + \frac{\alpha}{\alpha + \mu} \sum_{i \in C} m_i z_i^*(\alpha + \mu),
\end{equation}

where $z^*(\cdot) = (z^*_i(\cdot), i \in C)$ is given by $z_i^*(\alpha) = 1 - \sum_{j \in C} \alpha \phi_j^*(\alpha)$. Now, since $F$ satisfies $(FE_{ij})$ over $S$, we have in particular that

$$\alpha \phi_{i0}(\alpha) = \sum_{k \in C} \phi_{ik}(\alpha) q_{k0}, \quad i \in C.$$ 

But, from (4.2), we have that

$$\sum_{i \in C} m_i \phi_{ik}(\alpha) = \frac{m_k}{\alpha + \mu} (1 - z_k^*(\alpha + \mu)), \quad k \in C,$$

so that

$$\sum_{i \in C} m_i \alpha \phi_{i0}(\alpha) = \frac{1}{\alpha + \mu} \sum_{i \in C} m_i q_{i0} - \frac{1}{\alpha + \mu} \sum_{i \in C} m_i z_i^*(\alpha + \mu) q_{i0}.$$

This expression combines with (4.4) to give

$$\alpha \sum_{i \in C} m_i z_i^*(\alpha + \mu) + \sum_{i \in C} m_i z_i^*(\alpha + \mu) q_{i0} = (\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha) - \mu \sum_{i \in C} m_i (1 - a_i^f).$$
But, by Lemma 3.1, we have that \( h_i^F = a_i^F \) and so

\[
\alpha \sum_{i \in C} m_i z_i^*(\alpha + \mu) + \sum_{i \in C} m_i z_i^*(\alpha + \mu) q_{i0} = (\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha) - \mu \sum_{i \in C} m_i e_i^F,
\]

where \( e_i^F = 1 - h_i^F \). By letting \( \alpha \downarrow 0 \) in (4.5) we can now establish the required honesty of \( F^* \): using bounded convergence, we can take the limit under summation because \( m \) is convergent, and, by virtue of (4.1), \( \sum_{i \in C} m_i q_{i0} < \infty \). Since \( e_i^F = z_i(0+) \) and because \( z^*(\cdot) \) is continuous, we get \( \sum_{i \in C} m_i z_i^*(\mu) q_{i0} = 0 \). It follows that \( z^*(\mu) \) is identically zero, and hence that \( \sum_{j \in C} \alpha \phi_j^*(\alpha) = 1 \), when \( \alpha = \mu \). But equality here for any given \( \alpha > 0 \) is enough to ensure that \( F^* \) is honest.

The final statement of the theorem follows directly from Proposition 2 of Tweedie (1974).

Remarks. 1. The special case when \( F \) is assumed to be honest was dealt with in Pollett and Vere-Jones (1992); if \( F \) is honest, \( a_i^F = h_i^F = 1 \), for all \( i \), and so the necessary and sufficient condition, (4.1), reduces to

\[
\mu \sum_{i \in C} m_i = \sum_{i \in C} m_i q_{i0}.
\]

2. Under certain circumstances (4.6) follows directly from the premise that \( m \) is a \( \mu \)-invariant measure on \( C \) for \( Q \). For example, if \( m \) is a \( \mu \)-invariant measure on \( C \) for \( Q \), then a necessary and sufficient condition for (4.6) to hold is that

\[
\lim_{k \to \infty} \sum_{j=1}^{\infty} \sum_{i=k}^{\infty} m_i q_{ij} = 0,
\]

for, under this condition, the interchange of summation is justified in the following formal argument:

\[
\mu \sum_{j \in C} m_j = - \sum_{j \in C} \sum_{i \in C} m_i q_{ij} = - \sum_{i \in C} m_i \sum_{j \in C} q_{ij} = \sum_{i \in C} m_i q_{i0}
\]

(see for example) Section 3.7 of Knopp (1956)). Further, a necessary and sufficient condition for the double sum in (4.8) to be absolutely convergent is that

\[
\sum_{i \in C} m_i q_i < \infty,
\]

and so this condition is sufficient for (4.6). It follows that, when \( F \) is honest, (4.7) and (4.9) provide, respectively, a necessary and sufficient condition, and a simple sufficient condition, for a convergent \( \mu \)-invariant measure on \( C \) for \( Q \) to be a \( \mu \)-invariant measure on \( C \) for \( F \).

5. The identification of all single-exit \( Q \)-functions with a specified quasi-stationary distribution

We suppose that one is given a convergent \( \mu \)-subinvariant measure, \( m \), on \( C \) for \( Q \); it will not be necessary to assume that \( m \) is \( \mu \)-invariant for \( Q \). With the aid of
Reuter’s (1959) construction of all ‘single-exit’ $Q$-functions, we shall identify all such $Q$-functions for which $m$ is a $\mu$-invariant measure and, hence, a quasi-stationary distribution.

In his seminal work Reuter (1957) showed that the minimal $Q$-function is the unique $Q$-function if and only if $Q$ is regular, that is, the equations

$$
\sum_{j \in S} q_{ij} x_j = \xi x_i, \quad i \in S,
$$

have no bounded, non-trivial solution (equivalently, non-negative solution), $x$, for some (and then for all) $\xi > 0$. In addition, he established that when this condition fails there are infinitely many $Q$-processes, including infinitely many honest ones, and the dimension, $d$, of the space of bounded vectors, $x$, on $S$ satisfying (5.1) (a quantity which does not depend on $\xi$), determines the number of ‘escape routes to infinity’ available to the process. If, as we shall assume here, $d = 1$ (the ‘single-exit’ case), the form of the resolvent of any $Q$-function can be written down explicitly: if $P$ is such a $Q$-function, then (Reuter (1959)) either $P$ is the minimal $Q$-function or, otherwise, its resolvent, $\Psi(\cdot) = (\psi_j(\cdot), i, j \in S)$, must be of the form

$$
\psi_j(\alpha) = \phi_j(\alpha) + z_j(\alpha)y_j(\alpha), \quad i, j \in S, \quad \alpha > 0,
$$

where $y(\alpha) = (y_j(\alpha), j \in S)$ is of the form

$$
y_j(\alpha) = \frac{\eta_j(\alpha)}{c + \sum_{k \in S} \alpha \eta_k(\alpha)}, \quad j \in S, \quad \alpha > 0,
$$

c \geq 0, \quad \text{and} \quad \eta(\alpha) = (\eta_j(\alpha), j \in S) \text{ is a non-negative vector that satisfies}

$$
\sum_{k \in S} \eta_k(\alpha) < \infty, \quad \alpha > 0,
$$

and

$$
\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in S} \eta_k(\alpha) \phi_{kj}(\beta) = 0, \quad j \in S, \quad \alpha, \beta > 0.
$$

Further, $\Psi$ is honest if and only if $c = 0$. Notice that, since we have assumed that 0 is an absorbing state, $z_0 = 0$ and (5.5) can be written

$$
\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k \in C} \eta_k(\alpha) \phi_{kj}(\beta) = 0, \quad j \in C, \quad \alpha, \beta > 0,
$$

and

$$
\alpha \eta_0(\alpha) - \beta \eta_0(\beta) + (\alpha - \beta) \sum_{k \in C} \eta_k(\alpha) \beta \phi_{k0}(\beta) = 0, \quad \alpha, \beta > 0.
$$

It is important to realize that $\Psi$ is determined by $\eta$ and that, once $\eta$ is specified, a family of $Q$-functions (exactly one of which is honest) is obtained by varying $c$. Thus, the problem of identifying those $Q$-functions which satisfy a specified criterion, in our case, those $Q$-functions for which a specified measure is $\mu$-invariant on $C$, amounts to determining which choices of $\eta$ and $c$ are admissible.
Theorem 5.1. Suppose that \( Q \) is single-exit and that \( m \) is a convergent \( \mu \)-subinvariant measure on \( C \) for \( Q \). Then, there exists a \( Q \)-function for which \( m \) is \( \mu \)-invariant on \( C \) if and only if

\[
\sum_{i \in C} m_i q_{i0} \equiv \mu \sum_{i \in C} m_i.
\]

The resolvent, \( \Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S) \), of any \( Q \)-function for which \( m \) is a \( \mu \)-invariant measure on \( C \) must be of the form

\[
\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{z_j(\alpha)d_j(\alpha)}{(\alpha + \mu) \sum_{k \in C} m_k z_k(\alpha)}, \quad i, j \in S,
\]

with the interpretation that \( \Psi = \Phi \) if \( z \) is identically 0. The quantity \( d(\cdot) = (d_i(\cdot), i \in S) \) has the form

\[
d_i(\alpha) = m_i - \sum_{j \in C} m_j (\alpha + \mu) \phi_{ji}(\alpha), \quad i \in C,
\]

and

\[
d_0(\alpha) = \frac{e}{\alpha} - \sum_{i \in C} m_j (\alpha + \mu) \phi_{j0}(\alpha),
\]

where \( e \) satisfies

\[
\sum_{i \in C} m_i q_{i0} \equiv e \equiv \mu \sum_{i \in C} m_i.
\]

Conversely, if (5.6) holds, all \( Q \)-functions for which \( m \) is a \( \mu \)-invariant measure on \( C \) can be constructed through (5.7), (5.8) and (5.9), by varying \( e \) in the range given by (5.10). Exactly one of these is honest; this is obtained by setting \( e = \mu \sum_{i \in C} m_i \). And, exactly one satisfies \( (FE_{i0}) \) over \( C \); this is obtained by setting \( e = \sum_{i \in C} m_i q_{i0} \).

Proof. Let \( P \) be an arbitrary \( Q \)-function and let \( \Psi \) (given by (5.2)) be its resolvent. Then, the necessity of (5.6) follows immediately from Theorem 3.2. To complete the proof we shall first show that if \( m \) is \( \mu \)-invariant on \( C \) for \( P \), then \( \Psi \) must be of the form specified by (5.7). The sufficiency of (5.6), and the construction of all \( Q \)-functions for which \( m \) is \( \mu \)-invariant on \( C \), will then follow almost immediately.

Suppose, then, that \( m \) is \( \mu \)-invariant on \( C \) for \( P \) and, hence, for \( \Psi \). If \( P = F \) then there is nothing to prove. So, suppose that \( P \neq F \). Then, of necessity, \( z \) is not identically 0. On multiplying (5.2) by \((\alpha + \mu)m_i\) and summing over \( i \in C \), we find that

\[
m_j = \sum_{i \in C} m_i (\alpha + \mu) \phi_{ji}(\alpha) + (\alpha + \mu)y_j(\alpha) \sum_{i \in C} m_i z_i(\alpha),
\]

for all \( j \in C \). Hence, in view of (5.3), we require

\[
\frac{\eta_j(\alpha)}{c + \sum_{k \in S} \alpha \eta_k(\alpha)} = \frac{d_j(\alpha)}{(\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha)},
\]

where

\[
\eta_j(\alpha) = \sum_{i \in C} m_i \phi_{ji}(\alpha),
\]

and

\[
\frac{\eta_i(\alpha)}{c + \sum_{k \in S} \alpha \eta_k(\alpha)} = \frac{d_i(\alpha)}{(\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha)}.
\]
for all $j \in C$. Notice that, since $m$ is $\mu$-subinvariant on $C$ for $Q$, Proposition 1 of Tweedie (1974) implies that $m$ is $\mu$-subinvariant on $C$ for $F$ and so it is $\mu$-subinvariant on $C$ for $\Phi$. Thus $d_i(\alpha) \geq 0$, for all $i \in C$. Notice, also, that $\Sigma_{j \in C} d_j(\alpha)$ converges. Indeed, it is easy to show that

$$\alpha \sum_{j \in C} d_j(\alpha) = (\alpha + \mu) \sum_{j \in C} m_j z_j(\alpha) + (\alpha + \mu) \sum_{j \in C} m_j \phi_{j0}(\alpha) - \mu \sum_{j \in C} m_j;$$

Thus (5.4) holds good. To proceed, it will be necessary to deal separately with the two cases (i) where $d_j = 0$ for all $j \in C$, that is, $m$ is $\mu$-invariant on $C$ for $F$, and (ii) where $d_j > 0$ for at least one value of $j \in C$. If the latter holds, we see, from (5.11), that $\eta_i(\alpha) = K(\alpha)d_i(\alpha)$, for $j \in C$, where $K$ is a scalar function which satisfies

$$K(\alpha)\left( (\alpha + \mu) \sum_{k \in C} m_k z_k(\alpha) - \alpha \sum_{k \in C} d_k(\alpha) \right) = c + \alpha \eta_0(\alpha),$$

or equivalently, by (5.12),

$$K(\alpha)\left( \mu \sum_{k \in C} m_k - (\alpha + \mu) \sum_{k \in C} m_k \phi_{k0}(\alpha) \right) = c + \alpha \eta_0(\alpha).$$

We shall show that $K$ is a constant and that $\eta_0 = d_0$. We will then have established that $\Psi$ must be of the form (5.7). Using the resolvent equation for $\Phi$, in particular,

$$\phi_{ij}(\alpha) - \phi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in C} \phi_{ik}(\alpha) \phi_{kj}(\beta) = 0, \quad i, j \in C,$$

it is easy to show that

$$d_i(\alpha) - d_i(\beta) + (\alpha - \beta) \sum_{k \in C} d_k(\alpha) \phi_{ki}(\beta) = 0,$$

for all $i \in C$. But, since $\eta$ must satisfy (5.5), in particular (5.5a), we see, on substituting $\eta_i(\alpha) = K(\alpha)d_i(\alpha)$ and using (5.15), that $(K(\alpha) - K(\beta))d_i(\beta) = 0$. Thus $K$ is constant because $d_i$ is not identically 0 for at least one $j \in C$. Indeed, there is no loss of generality in setting $K = 1$, for this is equivalent to replacing $c$ by $c/K$ in (5.11). Thus $\eta_i = d_i$ for $j \in C$, and, from (5.13),

$$\alpha \eta_0(\alpha) = e - \sum_{j \in C} m_j (\alpha + \mu) \phi_{j0}(\alpha),$$

where $e = \mu \Sigma_{i \in C} m_i - c$. That is, $\eta_0 = d_0$. A straightforward calculation then shows that $\eta_0$ satisfies (5.5b), no matter what the value of $e$. In particular, we have that

$$ad_0(\alpha) - \beta d_0(\beta) + (\alpha - \beta) \sum_{k \in C} d_k(\alpha) \beta \phi_{k0}(\beta) = 0.$$
So, by Fatou’s lemma, we have that
\[ e \geq \sum_{i \in C} m_i \lim_{\alpha \to \infty} (\alpha + \mu)\alpha\phi_{i0}(\alpha) = \sum_{i \in C} m_i q_{i0}. \]
And, since we must have \( c \geq 0 \), we must also have that \( e \leq \mu \sum_{j \in C} m_j \).

Let us now deal with the case when, for each \( j \in C \), \( d_j \) is identically 0. From (5.11) we have that \( \eta_j = 0 \) and so (5.4) and (5.5a) are trivially satisfied. Now \( \eta_0 \) must be non-negative and satisfy (5.5b). But, on substituting \( \eta_j(\alpha) = 0 \), for \( j \in C \), this equation reduces to \( \alpha\eta_0(\alpha) - \beta\eta_0(\beta) = 0 \). Thus \( \alpha\eta_0(\alpha) \) and, hence, \( \alpha\psi_0(\alpha) \) are non-negative constants; let us write \( \alpha\psi_0(\alpha) = c_0 \), noting that \( c_0 \geq 1 \). We must demonstrate that this specification is consistent with (5.7), (5.9) and (5.10). Again using the resolvent equation (5.14), it is easy to show that \( z \) satisfies
\[ z_i(\alpha) - z_i(\beta) + (\alpha - \beta) \sum_{k \in C} \phi_{ik}(\alpha)z_k(\beta) = 0, \]
for all \( i \in C \), and so, on multiplying this equation by \( m_i \) and summing over \( i \in C \), we get
\[ (\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha) = (\beta + \mu) \sum_{i \in C} m_i z_i(\beta). \]
Thus \( (\alpha + \mu) \sum_{j \in C} m_j z_j(\alpha) \), and hence \( (\alpha + \mu) \sum_{j \in C} m_j \alpha\psi_{j0}(\alpha) \), do not depend on \( \alpha \); call these constant \( c_2 \) and \( c_3 \), respectively. Since \( \Phi \) satisfies the forward equations we have, in particular, that
\[ \alpha\psi_{i0}(\alpha) = \sum_{j \in C} \phi_{ij}(\alpha)q_{j0}, \quad i \in C. \]
Multiplying this equation by \( m_i \) and summing over \( i \in C \) shows that \( c_3 = \sum_{k \in C} m_k q_{k0} \), and so, from (5.12), \( c_2 = \mu \sum_{k \in C} m_k - \sum_{k \in C} m_k q_{k0} \). It follows that
\[ \frac{d_0(\alpha)}{(\alpha + \mu) \sum_{i \in C} m_i z_i(\alpha)} = \frac{e - \sum_{j \in C} m_j q_{j0}}{\mu \sum_{j \in C} m_j - \sum_{j \in C} m_j q_{j0}}, \]
a quantity which lies in the interval \([0, 1]\) if and only if \( e \) lies in the range specified by (5.10).

We shall now prove that (5.6) is a sufficient condition for there to exist a \( Q \)-function for which \( m \) is a \( \mu \)-invariant measure on \( C \) and, in so doing, specify the resolvent of all \( Q \)-functions for which \( m \) is \( \mu \)-invariant on \( C \). So, suppose that (5.6) holds. Then, let \( e \) be any number in the range specified by (5.10) and define \( \eta(\cdot) = (\eta_j(\cdot), j \in S) \) by \( \eta_j = d_j \), for \( j \in S \), where \( d \) is given by (5.8) and (5.9). It is immediate that \( \eta_j(\alpha) \geq 0 \), for all \( j \in C \), and that (5.4) holds. Also, by virtue of (5.15)
and (5.16), (5.5) holds. Define \( \Psi(\cdot) = (\psi_{ij}(\cdot), i, j \in S) \) by

\[
(5.17) \quad \psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{z_i(\alpha)\eta_j(\alpha)}{c + \sum_{k \in S} \alpha\eta_k(\alpha)}, \quad j \in S, \quad \alpha > 0,
\]

where \( c = \mu \sum_{i \in C} m_i - e \geq 0 \). In order to prove that \( \Psi \) is a \( Q \)-resolvent, it remains only to show that \( \eta_{i0}(\alpha)(= d_{i0}(\alpha)) \geq 0 \). But this follows because

\[
(\alpha + \mu) \sum_{j \in C} m_j \alpha \phi_{j0}(\alpha) = \sum_{k \in C} \sum_{j \in C} m_j (\alpha + \mu) \phi_{jk}(\alpha)q_{k0} \leq \sum_{k \in C} m_k q_{k0} \leq e.
\]

In view of (5.12) and (5.9), it is clear that (5.17) is the same as (5.7). It is then a matter of immediate verification that \( m \) is \( \mu \)-invariant on \( C \) for \( \Psi \).

The proofs of the concluding statements of the theorem are as follows. The \( Q \)-function, \( P \), whose resolvent is given by (5.7) is honest if and only if \( c = 0 \), that is if and only if \( e = \mu \sum_{i \in C} m_i \). Finally, we shall show that \( P \) satisfies \( (FE_{i0}) \) over \( C \) if and only if \( e = \sum_{i \in C} m_i q_{i0} \). If \( z \) is identically 0, then \( \Psi = \Phi \) and so there is nothing to prove. If \( z \) is not identically 0, then \( (FE_{i0}) \) holds for each \( i \in C \) if and only if

\[
\sum_{i \in C} \eta_i(\alpha) q_{i0} = \alpha \eta_0(\alpha),
\]

because it is easily verified that

\[
\sum_{k \in C} \psi_{ik}(\alpha) q_{k0} = \alpha \psi_{i0}(\alpha) + z_i(\alpha) \left( \sum_{k \in C} y_k(\alpha) q_{k0} - \alpha \gamma_0(\alpha) \right).
\]

But, by direct substitution we find that

\[
\alpha \eta_0(\alpha) - \sum_{i \in C} \eta_i(\alpha) q_{i0} = e - \sum_{k \in C} m_k q_{k0},
\]

and so the result follows.

**Remarks.**

1. If the condition that \( Q \) be single-exit is relaxed, condition (5.6) is sufficient, only, for the existence of a \( Q \)-function for which the specified measure is \( \mu \)-invariant on \( C \).

2. If \( \mu = 0 \), then (5.6) cannot be satisfied, because \( q_{i0} > 0 \) for at least one \( i \in C \). Thus if \( m \) is a convergent subinvariant measure on \( C \) for \( Q \), then there is no \( Q \)-function for which \( m \) is invariant on \( C \); contrast this with the case when \( C = S \) (see Pollett (1991a)).

3. A straightforward calculation shows that, if \( m \) is \( \mu \)-invariant on \( C \) for \( P \), then the quantity \( e \) given in the definition of \( d_{i0} \) has the value \( \mu \sum_{i \in C} m_i a_i^r \) and so, by Lemma 3.1, \( e = \mu \sum_{i \in C} m_i a_i^r \). Thus, in choosing \( e \), we are in fact specifying the likelihood that a Markov chain, with transition function \( P \) and initial distribution \( m \), ultimately reaches 0.

The final part of the theorem states that, under (5.6), there exists uniquely a \( Q \)-function which satisfies \( (FE_{i0}) \) over \( C \) and for which \( m \) is \( \mu \)-invariant on \( C \); note
that, by Theorem 3.1, the remaining forward equations are satisfied if and only if \( m \) is \( \mu \)-invariant on \( C \) for \( Q \). The existence of such a \( Q \)-function leads us to deduce the following result.

**Corollary 5.1.** If there exists a convergent \( \mu \)-subinvariant measure, \( m \), on \( C \) for \( Q \) which satisfies (5.6), then the bounds on \( \mu \) given by (3.3) and (3.7) are tight in that there always exists a \( Q \)-function, \( P \), such that \( \mu \sum_{i \in C} m_i a_i^p = \sum_{i \in C} m_i q_{i0} \).

As a final remark, observe that if (5.6) is satisfied with \( \sum_{i \in C} m_i q_{i0} = \mu \sum_{i \in C} m_i \), then the \( Q \)-function for which \( m \) is \( \mu \)-invariant is unique, honest, and satisfies \((\text{FE}_{i0})\) over \( C \). Thus, we have proved the following generalization of Corollary 1 of Pollett and Vere-Jones (1992).

**Corollary 5.2.** Suppose that \( Q \) is single-exit and that \( m \) is a convergent \( \mu \)-subinvariant measure on \( C \) for \( Q \). Then, there exists uniquely a \( Q \)-function, \( P \), for which \( m \) is \( \mu \)-invariant on \( C \) if and only if \( \sum_{i \in C} m_i q_{i0} = \mu \sum_{i \in C} m_i \). Its resolvent is given by (5.7), where \( d(\cdot) \) is specified by (5.8) and (5.9) with \( e = \mu \sum_{i \in C} m_i \). \( P \) is honest and it satisfies \((\text{FE}_{i0})\) over \( C \).

6. **Examples**

In this section we shall present some examples to illustrate our results. Since van Doorn's work on birth–death processes motivated the present study, we shall begin by explaining how his conditions for the existence of quasi-stationary distributions arise in the context of Theorem 4.1.

**Birth–death processes.** We shall adopt the usual notation in prescribing birth rates \((\lambda_i, i \geq 0)\) and death rates \((\mu_i, i \geq 0)\). Suppose that \( \lambda_i, \mu_i > 0 \), for \( i \geq 1 \), and \( \lambda_0 = \mu_0 = 0 \), and set

\[
q_{ij} = \begin{cases} 
\lambda_i, & \text{if } j = i + 1, \\
-(\lambda_i + \mu_i), & \text{if } j = i, \\
\mu_i, & \text{if } j = i - 1 \\
0, & \text{otherwise.}
\end{cases}
\]

Then, 0 is an absorbing state and \( C = \{1, 2, \ldots\} \) is irreducible for the minimal \( Q \)-function, \( F \), and hence for any \( Q \)-function. The classical theory of birth–death processes involves constructing a sequence of (orthogonal) polynomials using the equations for an \( x \)-invariant vector (see van Doorn (1991)). Define \((Q_i(\cdot), i \geq 1)\), where \( Q_i : \mathbb{R} \to \mathbb{R} \), by \( Q_1(x) = 1 \), \( \lambda_1 Q_2(x) = \lambda_1 + \mu_1 - x \), and, for \( i \geq 2 \),

\[
\lambda_i Q_{i+1}(x) - (\lambda_i + \mu_i) Q_i(x) + \mu_i Q_{i-1}(x) = -x Q_i(x).
\]

Now define \( \pi = (\pi_i, i \geq 1) \) by \( \pi_1 = 1 \) and

\[
\pi_i = \prod_{j=2}^{i} \frac{\lambda_{i-j-1}}{\mu_j}, \quad i \geq 2,
\]
and let \( m_i = \pi_i Q_i(x) \). It can be shown (van Doorn (1991)) that \( Q_i(x) > 0 \) for \( x \) in the range \( 0 \leq x \leq \lambda \), where \( \lambda(\geq 0) \) is the decay parameter of \( C \). Since \( \pi \) is a subinvariant measure for \( Q \), it follows, from Theorem 4.1 b(ii) of Pollett (1988), that, for each fixed \( x \) in the above range, \( m = (m_i, i \geq 1) \) is an \( x \)-invariant measure on \( C \) for \( Q \). Indeed, \( m \) is the unique \( x \)-invariant measure. Theorem 4.1 tells us that if \( m \) is convergent, that is \( \sum_{i=1}^{\infty} \pi_i Q_i(x) < \infty \), then \( m \) is a \( \mu \)-invariant measure on \( C \) for \( F \) if and only if

\[
\sum_{i=1}^{\infty} r_i(x) = 1,
\]

where \( r_i(x) = \mu_i^{-1} \pi_i x Q_i(x) \), a conclusion reached by van Doorn using direct methods. van Doorn's Theorem 3.2 can then be used to determine all the values of \( x \) for which (6.1) holds, at least under the condition that the minimal process is absorbed with probability 1, that is when

\[
\sum_{i=1}^{\infty} \frac{1}{\lambda_i \pi_i} = \infty.
\]

If, in addition, the series

\[
\sum_{i=1}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{j=i+1}^{\infty} \pi_j
\]

diverges then (6.1) holds for all \( x \) in \( (0, \lambda] \), while if it converges then (6.1) holds if and only if \( x = \lambda \). Proposition 2.1 then tells us that, in either case, \( r(x) = (r_i(x), i \geq 1) \) is a quasi-stationary distribution on \( C \) for \( F \). Indeed, because \( m \) is uniquely determined for each \( x \), all quasi-stationary distributions on \( C \) for \( F \) have been obtained under (6.2); if the series (6.3) converges, then there is only one, namely \( r(\lambda) \), while if (6.3) diverges, \( (r(x), 0 < x \leq \lambda) \) comprises a one-parameter family of quasi-stationary distributions.

A birth process with absorption. We shall illustrate the construction theory of Section 5 by considering a process taking values in \( S = \{0, 1, \cdots\} \) with a \( q \)-matrix \( Q = (q_{ij}, i, j \geq 0) \), given by

\[
q_{ij} = \begin{cases} 
\frac{1}{2} q_1, & \text{if } i = 1, j = 0, \text{ or } i = 1, j = 2, \\
q_i, & \text{if } i \geq 2, j = i + 1, \\
-q_i, & \text{if } j = i, \\
0, & \text{otherwise},
\end{cases}
\]

where, for \( i \geq 2 \), \( q_i > q_1 > q_0 = 0 \), and

\[
\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty.
\]

The probabilistic interpretation of these rates is obvious: if the minimal \( Q \)-process
starts in a state $i > 1$, it reaches infinity in a finite time. If it starts in state $i = 1$, then this scenario occurs with probability $\frac{1}{2}$; otherwise, the process is absorbed at 0.

Using the unique $\mu$-invariant measure for the pure-birth process without absorption, we can construct a convergent $\mu$-subinvariant measure on $C = \{1, 2, \cdots\}$. Let $m_1 > 0$ and, for $j > 1$, let

$$m_j = \frac{q_1 q_2 \cdots q_{j-1}}{(q_2 - q_1)(q_3 - q_1) \cdots (q_j - q_1)} m_1.$$ \hspace{1cm} (6.5)

Then it can be verified that $m = (m_j, j \geq 1)$ is a $\mu$-subinvariant measure on $C$ for $Q$, with $\mu = q_1$, and that

$$\sum_{j \in C} m_j = m_1 \prod_{j=2}^{\infty} \frac{q_j}{q_j - q_1}.$$ \hspace{1cm} (6.6)

Note that the specified measure is ‘almost’ $\mu$-invariant in that (2.1) holds with equality except for $j = 2$. It follows from (6.4) and (6.6) that $m$ is convergent.

We shall use Theorem 5.1 to construct the resolvents of all $Q$-functions for which $m$ is $\mu$-invariant; notice that condition (5.6) holds because

$$\sum_{i \in C} m_i q_{i0} = \frac{1}{2} m_1 q_1 < \mu \sum_{i \in C} m_i.$$  

The resolvent, $\Phi(\cdot) = (\phi_{ij}(\cdot), i, j \in S)$, of the minimal $Q$-function, obtained by solving (FE$_{ij}$), is given by

$$\phi_{ij}(\alpha) = \begin{cases} 
\frac{1}{q_i + \alpha} & \text{for } i = j \geq 0, \\
\frac{q_1}{2\alpha(q_1 + \alpha)}, & \text{for } i = 1, j = 0, \\
\frac{q_1 q_2 \cdots q_{j-1}}{2(q_1 + \alpha)(q_2 + \alpha) \cdots (q_j + \alpha)}, & \text{for } i = 1, j \geq 2, \\
\frac{q_i q_{i+1} \cdots q_{j-1}}{(q_i + \alpha) \cdots (q_j + \alpha)}, & \text{for } i \geq 2, j > i, \\
0, & \text{otherwise.}
\end{cases} \hspace{1cm} (6.7)$$

The quantity $d(\cdot) = (d_i(\cdot), i \in S)$ in (5.8) and (5.9) is given by

$$d_0(\alpha) = \frac{2e - m_1 q_1}{2\alpha}, \quad d_1(\alpha) = 0,$$

and, for $i \geq 2$,

$$d_i(\alpha) = \frac{(q_2 - q_1) \cdots (q_i - q_1)}{2(q_2 + \alpha) \cdots (q_i + \alpha)} m_i = m_1(q_1 + \alpha) \phi_{ii}(\alpha),$$
where \( e \) satisfies \( \frac{1}{2} m_1 q_1 \leq e \leq q_1 \sum_{i \in C} m_i \). Hence

\[
\alpha \sum_{j \in C} d_j(\alpha) = m_1(q_1 + \alpha)\{1 - \alpha(\phi_{10}(\alpha) + \phi_{11}(\alpha)) - z_1(\alpha)\},
\]

and, from (5.12), (6.7) and (6.8), it follows that

\[
(\alpha + \mu) \sum_{j \in C} m_j z_j(\alpha) = \alpha \sum_{j \in C} d_j(\alpha) + \mu \sum_{j \in C} m_j - (\alpha + \mu)m_1 \alpha \phi_{10}(\alpha)
\]

\[
= q_1 \sum_{j \in C} m_j - m_1(q_1 + \alpha)z_1(\alpha).
\]

Substituting these quantities in (5.7) we get

\[
\psi_{i0}(\alpha) = \phi_{i0}(\alpha) + \frac{(2e - m_1 q_1)z_1(\alpha)}{2\alpha \left\{ q_1 \sum_{j \in C} m_j - m_1(q_1 + \alpha)z_1(\alpha) \right\}}, \quad i \geq 0,
\]

\[
\psi_{i1}(\alpha) = \phi_{i1}(\alpha), \quad i \geq 0,
\]

and

\[
\psi_{ij}(\alpha) = \phi_{ij}(\alpha) + \frac{m_1(q_1 + \alpha)z_1(\alpha) \phi_{ij}(\alpha)}{q_1 \sum_{j \in C} m_j - m_1(q_1 + \alpha)z_1(\alpha)}, \quad i \geq 0, \quad j \geq 2.
\]

The probabilistic interpretation of the process with this resolvent is as follows: whenever the process hits infinity it either remains there or restarts in \( S \). The probability that it restarts at \( i \) is always proportional to \( a_i = \lim_{\alpha \to \infty} \alpha d_i(\alpha) \). It follows that \( a_0 = e - \frac{1}{2} m_1 q_1 \), \( a_1 = 0 \), \( a_2 = \frac{1}{2} m_1 q_1 \) and \( a_i = 0 \), for \( i \geq 3 \). Thus if, after hitting infinity, the process does return to \( S \), it is either absorbed at 0, or it starts in state 2. By Theorem 5.1 the process is honest if and only if \( e = q_1 \sum_{i \in C} m_i \), in which case it must always return to \( S \) after hitting infinity; the probabilities of restarting at 0 and 2 are, respectively,

\[
\frac{2q_1 \sum_{i \in C} m_i - m_1 q_1}{2q_1 \sum_{i \in C} m_i} \quad \text{and} \quad \frac{m_1}{2 \sum_{i \in C} m_i}.
\]

The unique process which satisfies (FE\(_{i0}\)) over \( C \) is obtained on setting \( e = \frac{1}{2} m_1 q_1 \). It is a dishonest process which has only one opportunity to be absorbed at 0; either it starts there, or it starts in state 1 and is absorbed on the first jump. If neither occurs, the process remains in \( \{2, 3, \ldots\} \) until it is eventually trapped at infinity.

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