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LIMITING CONDITIONAL AND CONDITIONAL INVARIANT DISTRIBUTIONS FOR THE POISSON PROCESS WITH NEGATIVE DRIFT

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Abstract

In this paper we study the conditional limiting behaviour for the virtual waiting time process for the queue M/D/1. We describe the family of conditional invariant distributions which are continuous and parametrized by the eigenvalues \( \lambda \in (0, \lambda_c] \), as it happens for diffusions. In this case, there is a periodic dependence of the limiting conditional distributions on the initial point and the minimal conditional invariant distribution is a mixture, according to an exponential law, of the limiting conditional distributions.

Keywords: Yaglom limit; quasi-stationary distributions; virtual waiting time process

AMS 1991 Subject Classification: Primary 60B10; 60K25

1. Introduction

Let \( (X'(t), t \geq 0) \) be the virtual waiting process associated with the queue M/D/1, starting from \( x > 0 \). Let \( T'_0 = \inf\{t \geq 0 : X'(t) = 0\} \) be the hitting time of 0. Before attaining 0 the process \( X' \) verifies

\[ X'(t) = x + N'(t) - \beta't, \quad \text{for } t < T'_0, \]

where \( N'(t) \) is a Poisson process with rate \( \lambda' > 0 \) and the service time is deterministic, taking value \( \beta' > 0 \). We shall study the subcritical case \( \lambda' < \beta' \). We reduce to the case \( \lambda = 1 \) by considering \( X(t) = X'(t/\lambda') \). In fact, for \( T_0 = \inf\{t \geq 0 : X(t) = 0\} \) the process \( X(t) \) verifies

\[ X(t) = x + N(t) - \beta t, \quad \text{for } t < T_0, \]

where \( \beta = \beta'/\lambda' > 1 \) and \( (N(t), t \geq 0) \) is a Poisson process with rate \( \lambda = 1 \).

Now, in Kyprianou [7] the virtual waiting time process is studied for the queues M/G/1 verifying that the Laplace transform, \( \psi(z) = \int_0^\infty e^{-zt} \, dG(t) \), of the service time distribution \( G \) is meromorphic in the \( \mathbb{C} \) extended plane. But this is not the case for a deterministic service because \( \psi(z) = e^{-z/\beta} \) has an essential singularity at \( \infty \).

We show in Theorem 2.1 of Section 2 the existence of limiting conditional distributions along subsequences for the process \( (X(t)) \). This limit depends on subsequences \( (t_n) \) due to the

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effect that the exit time \( T_0 \) from \((0, \infty)\) is of the form \( T_0 = (x + n)\beta \) for some \( n \in \mathbb{N} \), where \( x \) is the starting point for the process. In this result we show that all possible conditional limits as \( t \) tends to \( \infty \), excluding one case, are honest limits. This sensibility in the initial condition \( x \), which is periodic in \( x \), does not appear in the conditional limiting behaviour of queues studied by Kyprianou [7]. In fact he shows that the conditional limit \( \lim_{t \to \infty} P\{X(t) \in \cdot \mid T_0 > t\} \) exists and is independent of the initial condition.

In Section 3 we deal with \( \mathcal{L}^* \), the adjoint of the infinitesimal operator, which is a combination of a first order differential operator and a translation operator. We prove in Theorem 3.1 that conditional invariant distributions are absolutely continuous and have densities which are eigenfunctions of \( \mathcal{L}^* \). Then we show the existence of a one-dimensional parametrized family of eigenfunctions, verifying \( \mathcal{L}^* g_\lambda = -\lambda g_\lambda \) for \( \lambda \in \mathbb{R} \). In Theorem 3.2 we prove that the eigenfunctions are positive only if \( \lambda \leq \lambda_c = \beta \log \beta + 1 - \beta \), and they are integrable only if \( \lambda > 0 \). Hence they induce conditional invariant distributions only if \( \lambda \in (0, \lambda_c) \). This phenomenon is analogous to that in diffusions (see [2] and [9]) and birth–death chains (see [11]).

In Theorem 3.3 we show that the minimal conditional invariant distribution is an integral, with respect to an exponential law in the initial points, of the limiting conditional distributions. Finally, in Theorem 4.2 we characterize the weak limit distribution of the whole process, when conditioned to non-absorption (this limit corresponds to the \( h \)-process associated with the minimal conditional invariant distribution).

Our main tools are taken from the work of Pyke [12] on the supremum of a Poisson process, the results of Kijima [5] on skip-free Markov chains, and the use of the Wronskian associated with \( \mathcal{L}^* \).

There is a large amount of literature on the theory of limiting conditional distributions (or the Yaglom limit) and conditional invariant distributions (also called quasi-stationary distributions). Among others, we mention the pioneering works of Yaglom [15] on branching processes, Mandl [8] on diffusions, and Vere-Jones [14] and Seneta and Vere-Jones [13] on Markov chains. Some of the recent results for Markov chains are the work of Ferrari et al. [3], showing conditions for existence of conditional invariant distributions, and Kijima et al. [6] for birth–death chains.

2. Limiting conditional distributions

The purpose of this section is to characterize the possible weak limits, as \( t \) tends to \( \infty \), for the family of distributions \( \{P_x\{X(t) \in \cdot \mid T_0 > t\} : t > 0\} \). First, we study these limits along the following particular sequences \( t_n^a = (x + n - a)/\beta \), \( n \in \mathbb{N} \), for fixed \( a \in [0, 1) \). We are reminded that \( \lambda_c = \beta \log \beta + 1 - \beta \).

**Theorem 2.1.** For every \( a \in [0, 1) \), the following weak limit exists

\[
\mu_a(\cdot) = \lim_{n \to \infty} P_x\{X(t_n^a) \in \cdot \mid T_0 > t_n^a\},
\]

and it is an honest probability measure in \((0, \infty)\).

For \( a = 0 \) we have

\[
\mu_0(b) = (e^{\lambda_c/\beta} - 1) \frac{1}{\beta^b} \sum_{k=1}^{b} \frac{(-k)^{b-k}}{(b-k)!} e^k \text{ for } b \in \mathbb{N}^*, \quad (2.1)
\]
and for \( a \in (0, 1) \)

\[
\mu_a(b) = (1 - e^{-\lambda_c/\beta}) e^{a/\beta} \frac{e^{(b-k)} \sum_{k=0}^{[b]} (-a-k)^{[b]-k}}{\beta^{[b]} ([b] - k)!} e^k \quad \text{for } b \in a + \mathbb{N}.
\] (2.2)

**Proof of Theorem 2.1.** First we prove the result for \( a = 0 \). Denote by \((\tau_n, n \in \mathbb{N}^*)\) the sequence of stopping times where \((N(t), t > 0)\) jumps, i.e. \( \tau_n = \inf\{t > 0 : N(t) = n\} \). We have for fixed \( b \in \mathbb{N}^* \)

\[
P_x\{X(t_0^0) = b, T_0 > t_0^0\}
= P\left\{ \tau_1 \leq \frac{x}{\beta}, \ldots, \tau_n \leq \frac{x+n-1}{\beta} \left| N(t_0^0) = b + n \right\} P\{N(t_0^0) = b + n\}
= P\left\{ U^{(1)} \leq \frac{x}{\beta}, \ldots, U^{(n)} \leq \frac{x+n-1}{\beta} \right\} P\{N(t_0^0) = b + n\},
\]

where \( U^{(1)}, \ldots, U^{(n)}, \ldots, U^{(b+n)} \) are the order statistics of \( U_1, \ldots, U_n, \ldots, U_{b+n} \), which are independent and identically distributed random variables uniformly on \([0, t^0]\). Let \( V^{(i)} = U^{(i)}/t^0 \) \((i = 1, \ldots, b + n)\). Since \((V^{(1)}, \ldots, V^{(b+n)})\) and \((1 - V^{(b+n)}, \ldots, 1 - V^{(1)})\) have the same distribution, we get

\[
P \left\{ U^{(1)} \leq \frac{x}{\beta}, \ldots, U^{(n)} \leq \frac{x+n-1}{\beta} \right\}
= P \left\{ V^{(1)} \leq \frac{x}{x+n}, \ldots, V^{(n)} \leq \frac{x+n-1}{x+n}, \ldots, V^{(b+n)} \leq \frac{x+n+b-1}{x+n} \right\}
= P \left\{ \max_{1 \leq i \leq b+n} \left( \frac{i}{x+n} - V^{(i)} \right) \leq \frac{b}{x+n} \right\}
= \frac{x}{(x+n)^{b+n}} \sum_{j=0}^{b} \binom{b+n}{j} (j-b) (x+b+n-j)^{b+n-j-1},
\]

where the last equality follows from [12]. Hence

\[
P_x\{X(t_0^0) = b, T_0 > t_0^0\} = \frac{x e^{x+n-t_0^0}}{\sqrt{2\pi n^3} \beta^{b+n}} \sum_{k=1}^{b} \frac{(-k)^{b-k} e^k}{(b-k)!} \gamma_n(k),
\]

\[
P_x\{T_0 > t_0^0\} = \frac{x e^{x+n-t_0^0}}{\sqrt{2\pi n^3} \beta^m} \sum_{k=1}^{\infty} \frac{1}{\beta^m} \sum_{m=1}^{m} \frac{(-k)^{m-k} e^k}{(m-k)!} \gamma_n(k),
\] (2.3)

where \( \gamma_n(k) = \sqrt{2\pi n^3} (x+n+k)^{n+k-1}/((n+k)! e^{x+n+k}) \), which by Stirling’s formula verifies \( 0 \leq \gamma_n(k) \leq 1 \) and for each \( k \in \mathbb{N} \), \( \lim_{n \to \infty} \gamma_n(k) = 1 \). Therefore,

\[P_x\{X(t_0^0) = b \mid T_0 > t_0^0\} = P_n(b)/Q_{n,x}(1/\beta),\]

where

\[P_n(b) = \frac{1}{\beta^b} \sum_{k=1}^{b} \frac{(-k)^{b-k} e^k}{(b-k)!} \gamma_n(k) \quad \text{and} \quad Q_{n,x}(1/\beta) = \sum_{m=1}^{\infty} \frac{1}{\beta^m} \sum_{k=1}^{m} \frac{(-k)^{m-k} e^k}{(m-k)!} \gamma_n(k).\]
We shall prove the result for \( x \in \mathbb{N}^* \). In this case the process \( X(t_n^0) : n \in \mathbb{N} \) is a skip-free Markov chain, with transition probabilities \( p_{ij} = a_{j-i+1} \), \( j \geq i - 1 \) where \( a_k = e^{-(1/\beta)}((1/\beta)^k/k!) \), \( k \geq 0 \). The conditioning expression is \( \{ T_0 > t_n^0 \} = \{ X(t_k^0) > 0, k = 0, \ldots, n \} \), therefore, the desired limit is the conditional limiting distribution for this chain. From the results of [5] this limit exists, because in our case \( m = -a_0 + \sum_{k=1}^{\infty} (k-1) a_k < 0 \) (see Section 4 in [5]), and it is an honest distribution on \( \mathbb{N}^* \).

Since \( P_n(b) \) converges to \( (1/\beta^b) \sum_{k=1}^{b} ((-k)^{b-k} e^k)/(b-k)! \) we deduce that \( Q_n,x(1/\beta) \) converges to \( \sum_{b=1}^{\infty} (1/\beta^b) \sum_{k=1}^{b} ((-k)^{b-k} e^k)/(b-k)! \), where we have used the fact that the limit is honest. Let us compute this last series.

It is easy to see that there exists \( r_0 > 0 \) such that the series \( \sum_{b=1}^{\infty} (\sum_{k=1}^{b} ((-k)^{b-k} e^k)/(b-k)!)z^b \) converges on \( \{ z \in \mathbb{C} : |z| < r_0 \} \) to \( G(z) = (z e^{1-z})/(1 - z e^{1-z}) \). Let \( D_0(1) = \{ z \in \mathbb{C} : |z| < 1 \} \). Since \( G(z) \) is analytic on \( D_0(1) \), it follows that for each \( z \in D_0(1) \),

\[
\frac{z e^{1-z}}{1 - z e^{1-z}} = \sum_{b=1}^{\infty} \left( \sum_{k=1}^{b} \frac{(-k)^{b-k} e^k}{(b-k)!} \right) z^b
\]

therefore \( \lim_{n \to \infty} Q_{n,x}(1/\beta) = G(1/\beta) = \beta e^{(1/\beta)-1} \). Then the result holds for \( x \in \mathbb{N}^* \) and \( a = 0 \).

Now assume that \( x > 0 \) is not an integer. First of all we shall prove that the family \( \{ P_x \{ X(t_n^0) \in \cdot \mid T_0 > t_n^0 \}, n \in \mathbb{N}^* \} \) is tight in \( (0, \infty) \), First note that these measures are concentrated on \( \mathbb{N}^* \), then it suffices to prove that the family of measures is tight on \( [0, \infty) \). For this purpose, consider \( \theta \in (1, \beta) \). From (2.3) we deduce that

\[
E_x(\theta^X(t_n^0) \mid T_0 > t_n^0) = \frac{Q_{n,x}(\theta/\beta)}{Q_{n,x}(1/\beta)}.
\]

Notice that

\[
P_x \left\{ T_0 > \frac{x+n}{\beta} \right\} \leq P_x \left\{ T_0 > \frac{x+n}{\beta} \right\} \leq P_{[x]} \left\{ T_0 > \frac{[x]+n-1}{\beta} \right\},
\]

and

\[
P_x \left\{ T_0 > \frac{x+n}{\beta} \right\} \geq P_x \left\{ N(x/\beta) = 1, T_0 > \frac{x+n}{\beta} \right\} = \frac{x e^{-x/\beta}}{\beta} P_1 \left\{ T_0 > \frac{1+(n-1)}{\beta} \right\},
\]

where \( [x] \) is the smallest integer greater than \( x \). Therefore we deduce that there is a finite constant \( C(x) \), depending only on \( x \), such that

\[
E_x(\theta^X(t_n^0) \mid T_0 > t_n^0) \leq C(x) \frac{Q_{n-1,[x]}(\theta/\beta)}{Q_{n-1,1}(1/\beta)},
\]

which is bounded in \( n \), because \( \frac{Q_{n-1,[x]}(\theta/\beta)}{Q_{n-1,1}(1/\beta)} \) converges to \( G(\theta/\beta)/G(1/\beta) \). Thus, the tightness on \( [0, \infty) \) follows. From (2.3) one easily obtains that the only possible weak limit of \( \{ P_x \{ X(t_n^0) \in \cdot \mid T_0 > t_n^0 \}, n \in \mathbb{N} \} \) is \( \mu_0 \) given by (2.1). Also we obtain

\[
\lim_{n \to \infty} Q_{n,x}(1/\beta) = G(1/\beta).
\]
Let us now treat the case \( a \in (0, 1) \). To prove (2.2) notice that \( T_0 > t_n^a \Leftrightarrow T_0 > t_{n-1}^0 \). Then,

\[
P_x \{ X(t_n^a) = \ell + a \mid T_0 > t_n^a \} = \sum_{k=1}^{\ell+1} P[N(t_n^a) - N(t_{n-1}^0) = \ell + 1 - k] P_x \{ X(t_n^0) = k \mid T_0 > t_{n-1}^0 \}.
\]

By using (2.1) we find that

\[
\lim_{n \to \infty} P_x \{ X(t_n^a) = \ell + a \mid T_0 > t_n^a \} = \left( \beta e^{(1/\beta) - 1} - 1 \right) \frac{\beta^{\ell+1}}{\beta^j+1} \sum_{j=1}^{\ell+1} e^j \sum_{k=j}^{\ell+1} \frac{(1-a)\ell+1-k(-j)^k-j}{(\ell + 1 - k)(k - j)!}.
\]

Make the change of variables \( k' = k - j \), use the Binomial Newton’s formula and take \( j' = j - 1 \), to arrive at equality (2.2), with \( e = [b] \).

We remark that \( \lim_{n \to \infty} \mu_a(a+n) = \mu_0(1+n) \) for all \( n \in \mathbb{N} \). Now take \( t_n^{-a} = (x+n+a)/\beta \), by using the equivalence \( T_0 > t_n^{-a} \Leftrightarrow T_0 > t_n^0 \) it is easy to show that the following limit exists:

\[
\mu_{-a}(b) = \lim_{n \to \infty} P[X(t_n^{-a}) = b \mid T_0 > t_n^{-a}],
\]

for \( b \in \mathbb{N}^* - a \), that it is an honest distribution and is given by

\[
\mu_{-a}(b) = e^{-a/\beta} \beta e^{(1/\beta) - 1} - 1 \frac{\beta^{|b|+1}}{\beta^{|b|+1}} \sum_{k=1}^{|b|+1} \frac{e^k(a - k)[b] + 1 - k}{(|b|+1 - k)!}.
\]

Also we have \( \lim_{n \to 0} \mu_{-a}(n - a) = \mu_0(n) \) for all \( n \in \mathbb{N}^* \). We can also prove the existence of \( \mu_a \) for \( a \in \mathbb{R} \), which is periodic in \( a \); in fact we have \( \mu_a = \mu_a' \) if \( a \equiv a' \) (mod 1).

**Corollary 2.2.** For any \( a \) and any \( k \in \mathbb{N} \) the following ratio limit exists:

\[
\frac{P_x \{ T_0 > t_{n+k}^a \}}{P_x \{ T_0 > t_n^a \}} \xrightarrow{n \to \infty} e^{-(\lambda_c/\beta)k}.
\]

This implies

\[
-\frac{1}{t} \log P_x \{ T_0 > t \} \xrightarrow{n \to \infty} \lambda_c.
\]

**Proof.** The first relation (2.5) follows directly from (2.3) and (2.4). Now (2.5) implies

\[
-\lim_{n \to \infty} \left( \frac{n}{\beta} \right)^{-1} \log P_x \left\{ T_0 > \frac{x}{\beta} + \frac{n}{\beta} \right\} = \beta \log \beta + 1 - \beta.
\]

Then (2.6) follows by taking \( x/\beta + n/\beta \leq t < x/\beta + (n+1)/\beta \).

**Corollary 2.3.** The only weak limit points, as \( t \) tends to \( \infty \), of the family of probability measures \( \{ P_x \{ X(t) \in \cdot \mid T_0 > t \} : t > 0 \} \) is \( \{ \mu_a : a \in (0, 1) \} \cup \{ \mu_* \} \), where \( \mu_* \) is a probability measure on \( \mathbb{N} \), giving positive mass to 0. More precisely \( \mu_*(0) = 1 - e^{1-1/\beta - \log \beta} \), and \( \mu_*(A \mid \mathbb{N}^*) = \mu_0(A) \).
Proof. As in the previous theorem \( \{P_x[X(t) \in \cdot \mid T_0 > t] : t > 0\} \) is a tight family of distributions on \([0, \infty)\). Now, take \( \mu \) any such limit distribution and a sequence \( t_k \uparrow \infty \) such that \( \lim_{k \to \infty} P_x[X(t_k) \in \cdot \mid T_0 > t_k] = \mu(\cdot) \). Since we have already analysed the case \( t_k = (x + n_k - a_k)/\beta \) when \( a_k \) is ultimately constant, we can assume \( a_k \in (0, 1) \) converging to \( a \in [0, 1] \). In the case \( a \in (0, 1) \) it is not hard to prove that \( \mu = \mu_a \). Therefore, in what follows, we assume \( a = 0 \). Take \( \delta \in (0, 1) \) and define \( \ell_k = (x + n_k - 1)/\beta \), \( u_k = (x + n_k)/\beta \).

We have for large \( k \)

\[
P_x[X(t_k) \leq \delta \mid T_0 > t_k] = P_x[X(\ell_k) = 1, N(t_k) - N(\ell_k) = 0 \mid T_0 > \ell_k]
\]

which converges to \( \mu_0(1)e^{-\lambda/\beta} = 1 - e^{-\lambda/\beta} \).

For the rest of the proof we assume \( m \in \mathbb{N}^* \). As before take \( \delta \in (0, 1) \). We have for large \( k \)

\[
P_x[X(t_k) \in (m - \delta, m + \delta) \mid T_0 > t_k] = P_x[X(t_k) = m + a_k \mid T_0 > t_k]
= P_x[X(u_k) = m \mid T_0 > t_k] + R_k
= P_x[X(u_k) = m, T_0 > u_k] + R_k
= P_x[X(u_k) = m \mid T_0 > u_k]P_x[T_0 > u_k] + R_k,
\]

where \( 0 \leq R_k \leq P_x[N(u_k) - N(t_k) \geq 1] \) converges to 0 when \( k \) tends to \( \infty \). The result now follows from the previous corollary, and Theorem 2.1.

3. Conditional invariant distributions

The infinitesimal generator associated with the Markov process \( \{X(t), t \geq 0\} \) is given by \( \mathcal{L}f(x) = -\beta f'(x) + f(x + 1) - f(x) \), for smooth functions \( f \) vanishing at 0.

Let \( \mathcal{L}^* \) be the adjoint operator of \( \mathcal{L} \). Then, for \( g \) a differentiable density (with respect to the Lebesgue measure), we have

\[
\mathcal{L}^*g(x) = \beta g'(x) + g(x - 1)1_{\{x \geq 1\}} - g(x).
\]

We note that a probability measure \( \mu \) defined on \((0, \infty)\) is conditional invariant if it verifies

\[
P_\mu[X(t) \in A \mid T_0 > t] = \mu(A) \quad \text{for any Borel subset } A \subseteq (0, \infty) \text{ and any } t \geq 0,
\]

or equivalently if

\[
E_\mu(f(X(t)) \mid T_0 > t) = \int f \, d\mu \quad \text{for any bounded Borel function } f \text{ and any } t \geq 0.
\]

By applying the Markov semigroup property we find that

\[
P_\mu[T_0 > t + s] = P_\mu[T_0 > t]P_\mu[T_0 > s].
\]

Then there exists \( \lambda > 0 \) such that \( P_\mu[T_0 > t] = e^{-\lambda t} \) for all \( t \geq 0 \), so \( T_0 \) is exponentially distributed when the initial distribution is \( \mu \). Hence \( \mu \) is conditional invariant if and only if for some \( \lambda > 0 \) it verifies

\[
E_\mu(f(X(t)), T_0 > t) = e^{-\lambda t} \int f \, d\mu \quad \text{for any bounded Borel function } f \text{ and } t \geq 0,
\]

because this equation implies, by taking \( f \equiv 1 \), that \( P_\mu[T_0 > t] = e^{-\lambda t} \).
Now we will show that the conditional invariant measures appear as eigenfunctions of $\mathcal{L}^*$ with strictly negative eigenvalues. As we state in the Introduction for countable state Markov chains this was proved in [11]. For diffusions on compact regions this is easy to show from the discreteness of the spectrum, for one-dimensional diffusions this result also holds (in this context see [10]). For continuous state space one of the main problems is to show that conditional invariant distributions are absolutely continuous functions, which in this case becomes harder because $\mathcal{L}^*$ is not a differential operator, in fact it possesses a translation part.

**Theorem 3.1.** A probability measure $\mu$ on $(0, \infty)$ is conditional invariant if and only if it is absolutely continuous with respect to the Lebesgue measure and its derivative $g = d\mu/dx$ is $C^1$, except perhaps at $x = 1$, and verifies

$$\mathcal{L}^* g = -\lambda g,$$

for some $\lambda > 0$.

**Proof.** We first show that the condition is necessary. Then, consider a conditional invariant distribution. Denote by $\lambda > 0$ the exponential rate of the exit time starting from $\mu$. Now pick $f$, a $C^2$ function, such that $|f'|, |f''|$ are bounded and $f(0) = 0$. Relation (3.2) implies that

$$\int \mu(dx) E_x(f(X(t)))1_{T > t} - e^{-\lambda t} f(x) = 0.$$

Since the process is not stopped at time $t$ if $x > \beta t$ we find that

$$\frac{1}{t} \int_{\beta t}^\infty \mu(dx) E_x(f(X(t))) - e^{-\lambda t} f(x) + \frac{1}{t} \int_0^{\beta t} \mu(dx) E_x(f(X(t)))1_{T > t} - e^{-\lambda t} f(x) = 0.$$

(3.3)

The first integral, which we denote by $I_1^f$, is

$$I_1^f = \int_{\beta t}^\infty \mu(dx) \frac{f(x - \beta t) - e^{-\lambda t} f(x)}{t} \mathbb{P}(N(t) = 0)$$

$$+ \int_{\beta t}^\infty \mu(dx) (f(x + 1 - \beta t) - e^{-\lambda t} f(x)) \mathbb{P}(N(t) = 1) = R_t + \frac{R_t}{t},$$

where $R_t/t \leq 2\|f\| \mathbb{P}(N(t) \geq 2)/t \to t \to 0^+ 0$.

To analyse the first term of $I_1^f$ we expand $f$ to get

$$\frac{f(x - \beta t) - e^{-\lambda t} f(x)}{t} = \lambda f(x) - \beta f'(x) + Q_t,$$

with $|Q_t| \leq tD_1$ (we use the boundedness of $f''$) for some constant $D_1$. Then,

$$I_1^f \to t \to 0^+ \int_0^\infty (\lambda f(x) - \beta f'(x)) d\mu(x) + \int_0^\infty (f(x + 1) - f(x)) d\mu(x)$$

$$= \int_0^\infty (\mathcal{L} - \lambda) f(x) d\mu(x).$$

Let us analyse the second integral in (3.3) which we denote by $I_1^f$. First observe that $f(0) = 0$ and $f'$ bounded implies that for some constant $D_2 > 0$ we have $|f(x)| \leq D_2 t$ for every $x \leq \beta t$. Hence

$$|I_1^f| \leq \int_0^{\beta t} \|f\| \mathbb{P}(N(t) \geq 1) d\mu(x) + D_2 \int_0^{\beta t} d\mu(x) \to t \to 0^+ 0.$$
We conclude that for any $C^2$ function, with $f'$, $f''$ bounded and $f(0) = 0$, it is verified that

$$\int_0^\infty (\mathcal{L} - \lambda) f(x) \, d\mu(x) = 0. \quad (3.4)$$

This implies that

$$-\beta \int f'(x) \, d\mu(x) = \int f(x) \, d\nu(x), \quad (3.5)$$

where $\nu$ is a signed finite measure verifying

$$d\nu(x) = (1 - \lambda) \, d\mu(x) - d\mu(x - 1)1_{[x>1]}.$$  

Assume $f$ also verifies $\lim_{x \to \infty} f(x) = 0$. By integrating by parts we find

$$\int f \, d\nu = \int f'(x)\nu[x, \infty) \, dx.$$

From equality (3.5) we obtain

$$-\beta \int f'(x) \, d\mu(x) = \int f'(x)\nu[x, \infty) \, dx,$$

for every $f$ fulfilling the above conditions. We deduce that $\mu$ is absolutely continuous and its derivative $g = d\mu/dx$ verifies $-\beta g(x) = \nu[x, \infty) \, dx$ almost everywhere. From (3.4), (3.5) we find that $g$ verifies the relation

$$-\beta g(x) = \int_x^\infty (1 - \lambda) g(\xi) \, d\xi - \int_x^\infty g(\xi - 1)1_{[\xi>1]} \, d\xi.$$  

Since the right-hand side is continuous in $x$ the function $g$ has a continuous version. Moreover $g$ is differentiable, except perhaps at $x = 1$, and it verifies

$$-\beta g'(x) = -(1 - \lambda)g(x) + g(x - 1)1_{[x>1]}.$$  

Hence $\mathcal{L}^* g = -\lambda g$.

Reciprocally, we shall show that any probability measure $\mu$ with a density function $g$ verifying $\mathcal{L}^* g = -\lambda g$ for some $\lambda > 0$ is conditional invariant. In order to prove this claim, we take a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ with the extra condition $f(0^+) = 0$ (where $f(0^+) = \lim_{x \to 0} f(x)$). We shall show the relation

$$\lim_{s \to 0} \frac{1}{s} \int g(x)(E_x(f(X_s), T_0 > s) - f(x)) \, dx = -\lambda \int g(x) f(x) \, dx. \quad (3.6)$$

The left integral will be decomposed into two pieces: $\int = \int_0^{\beta s} + \int_0^{\infty}$. Concerning the first term, observe that for $x < \beta s$

$$|E_x(f(X_s), T_0 > s)| \leq \|f\|_{\infty} P_x\{T_0 > s\} \leq \|f\|_{\infty} P\{N_{(x/\beta)} > 1\} \leq \|f\|_{\infty} (1 - e^{-x/\beta}) \leq A\|f\|_{\infty} x,$$
for some finite constant \( A \). Then, by using \( f(0^+) = 0 \), we get
\[
\lim_{s \searrow 0} \frac{1}{s} \int_0^{\beta s} g(x)(E_x(f(X_s), T_0 > s) - f(x)) \, dx = 0. \tag{3.7}
\]

Let us analyse the second term \( \int_{\beta s}^{\infty} \). Observe that
\[
E_x(\cdot, T_0 > s) = E_x(\cdot) \quad \text{for } x > \beta s. \tag{3.8}
\]

Also we have
\[
E_x(f(X_s) - f(x)) = (f(x - \beta s) - f(x)) e^{-s} + (f(x + 1 - \beta s) - f(x)) s e^{-s} + R(s), \tag{3.9}
\]

with \( |R(s)| \leq B \| f \|_{\infty} s^2 \) for some finite constant \( B \). Now
\[
\lim_{s \searrow 0} \frac{1}{s} \int_0^{\infty} g(x)(f(x + 1 - \beta s) - f(x)) s e^{-s} \, dx = \int_0^{\infty} g(x)(f(x + 1) - f(x)) \, dx. \tag{3.10}
\]

On the other hand
\[
\int_{\beta s}^{\infty} g(x)(f(x - \beta s) - f(x)) \, dx = \int_0^{\infty} g(x + \beta s) f(x) \, dx - \int_{\beta s}^{\infty} g(x) f(x) \, dx
\]
\[
= \int_{\beta s}^{\infty} (g(x + \beta s) - g(x)) f(x) \, dx + \int_0^{\beta s} g(x + \beta s) f(x) \, dx.
\]

Then
\[
\lim_{s \searrow 0} \frac{1}{s} \int_{\beta s}^{\infty} g(x)(f(x - \beta s) - f(x)) e^{-s} \, dx
\]
\[
= \lim_{s \searrow 0} e^{-s} \left( \int_{\beta s}^{\infty} \frac{(g(x + \beta s) - g(x))}{s} f(x) \, dx + \frac{1}{s} \int_0^{\beta s} g(x + \beta s) f(x) \, dx \right) \tag{3.11}
\]
\[
= \int_0^{\infty} \beta g'(x) f(x) \, dx.
\]

To establish the last equality we have used the bounds \( |\beta g'(\xi)| \leq (1 + \lambda) g(\xi) + g(\xi - 1) \mathbf{1}_{\{\xi \geq 1\}} \) and the hypothesis condition \( f(0^+) = 0 \). By using the equality \( \beta g'(x) = (1 - \lambda) g(x) - g(x - 1) \mathbf{1}_{\{x \geq 1\}} \) we find
\[
\int_0^{\infty} \beta g'(x) f(x) \, dx = -\lambda \int_0^{\infty} g(x) f(x) \, dx + \int_0^{\infty} g(x)(f(x) - f(x + 1)) \, dx. \tag{3.12}
\]

From relations (3.7)–(3.12) we conclude (3.6).

Now take a bounded continuous real function \( f : \mathbb{R}_+^* \rightarrow \mathbb{R} \). We define, for a fixed \( t > 0 \), the function
\[
f_t(x) = E_x(f(X(t)), T_0 > t), \quad x \in \mathbb{R}_+^*.
\]
We will prove that $f_t$ is a continuous function in $x$, and it verifies $f_t(0^+) = 0$. The last relation follows from the fact that for $x/\beta \leq t$

$$|f_t(x)| \leq \|f\|_\infty P_x\{T_0^x > t\} \leq \|f\|_\infty P\{N(x/\beta) \geq 1\} \xrightarrow{x \downarrow 0} 0.$$ 

Let us prove the continuity of $f_t$. Let $x > 0$ and $0 < x' < x$. We denote by $T_0^x$ and $T_0^{x'}$ the absorption times when starting from $x$ and $x'$ respectively. We have

$$f_t(x) - f_t(x') = E(f(x + N(t) - \beta t), T_0^x > t) - E(f(x' + N(t) - \beta t), T_0^{x'} > t)
= E(f(x + N(t) - \beta t) - f(x' + N(t) - \beta t), T_0^{x'} > t)
+ E(f(x + N(t) - \beta t), T_0^x \leq t < T_0^{x'}).$$

Concerning the last term of the sum at the right-hand side, we have

$$E(f(x + N(t) - \beta t), T_0^{x'} \leq t < T_0^x) \leq \|f\|_\infty P(T_0^{x'} \leq t < T_0^x).$$

Define $\Delta_t = \min\{X_u, u < t\}$. It is easy to see that $\Delta_t = (\min\{x+k-1-\beta \tau_k : \tau_k \leq t\}) \wedge X(t)$, where $(\tau_k)$ are the jump times for the Poisson process. Now $T_0^{x'}(w) > t$ implies $\Delta_t(w) > 0$ almost everywhere. Then for any $x' > x - \Delta_t(w)$ we have $T_0^{x'}(w) > t$. From the monotone convergence theorem we deduce that $\lim_{x' \uparrow x} P(T_0^{x'} \leq t < T_0^x) = 0$. We have shown that the left term in (3.14) converges to 0 when $x' \nearrow x$. Concerning the first term of the sum in (3.13) we get the following from the dominated convergence theorem:

$$|E(f(x + N(t) - \beta t) - f(x' + N(t) - \beta t), T_0^x > t) |
\leq E(|f(x + N(t) - \beta t) - f(x' + N(t) - \beta t)|) \xrightarrow{x \to x} 0.$$

Therefore for any fixed $t > 0$ the function $f_t(\cdot)$ is left continuous. By a similar argument it is shown that $f_t(\cdot)$ is right continuous.

To finish the proof define $u(t) = \int g(x) f_t(x) \, dx$. From (3.6) and the Markov property we deduce that $u'(t) = -\lambda u(t)$, for all $t \geq 0$. On the other hand

$$u(t) \xrightarrow{t \downarrow 0} u(0) = \int g(x) f(x) \, dx.$$

We conclude that

$$\int g(x) E_x(f(X(t))), T_0 > t \, dx = e^{-\lambda t} \int g(x) f(x) \, dx.$$

Using the monotone convergence theorem to approach $f \equiv 1$, we find that the probability measure $\mu$ with density $g$ verifies $P_\mu\{T_0 > t\} = e^{-\lambda t}$. Then $\mu$ is conditional invariant.

The next result characterizes, according to the Theorem 3.1, the conditional invariant distributions.

**Theorem 3.2.** Let $\lambda \in \mathbb{R}$ and denote $\theta = (1 - \lambda)/\beta$. The function

$$g_\lambda(x) = \frac{e^{\theta(x-k)}}{\beta^k} \sum_{j=0}^{k} \frac{(\beta e^\theta)^j (k-x-j)^{k-j}}{(k-j)!} 1_{[k,k+1]}(x)$$

satisfies $dx$-almost everywhere.
\[ \beta g'(x) - g(x) + g(x - 1)1_{\{x \geq 1\}} = -\lambda g(x). \]  

(3.17)

This solution is the unique continuous solution of (3.1), up to a multiplicative constant. \( g_\lambda \) is a positive function if and only if \( \lambda \leq \lambda_c = \beta \log \beta + 1 - \beta \) and it is integrable if and only if \( \lambda > 0 \). Therefore it is positive and integrable if and only if \( 0 < \lambda \leq \lambda_c \). For \( \lambda > 0 \) we have

\[ \int_0^\infty g_\lambda(x) \, dx = \frac{\beta}{\lambda}. \]

In the positive and integrable case the eigenfunctions diverge exponentially fast, more precisely

\[ g_\lambda(x) \geq \exp((\lambda' - \lambda)/\beta) x g_{\lambda'}(x) \quad \text{for } 0 < \lambda < \lambda' \leq \lambda_c. \]  

(3.18)

Furthermore, if \( \mu^\lambda \) is the probability measure with density function \( (\lambda/\beta) g_\lambda \) for \( \lambda \in (0, \lambda_c] \), then

\[ P_{\mu^\lambda}(T_0 > t) = e^{-\lambda t}, \quad \text{for all } t \geq 0, \]

and \( \mu^\lambda \) is conditional invariant

\[ E_{\mu^\lambda}(f(X(t)) \mid T_0 > t) = \int f \, d\mu^\lambda, \quad \text{for all } t \geq 0. \]

Proof. We shall write \( g \) instead of \( g_\lambda \) where no confusion is possible. For \( 0 \leq x < 1 \)

\[ \beta g'(x) = (1 - \lambda) g(x) \]

and therefore \( g(x) = e^{\theta x} \), where we have fixed \( g(0) = 1 \). We also consider \( g(x) = 0 \) for \( x < 0 \). It follows by induction that for \( x \in [k, k + 1) \)

\[ g(x) = e^{\theta(x-k)} \sum_{j=0}^k \frac{g(j)}{(k-j)!} \left( \frac{k-x}{\beta} \right)^{k-j}. \]

Since \( g \) is a continuous function we deduce the recursion formula

\[ g(k + 1) = e^\theta \sum_{j=0}^k \frac{g(j)}{(k-j)!} \left( -\frac{1}{\beta} \right)^{k-j} \quad \text{for } k \geq 0. \]

This recursion has a unique solution given by

\[ g(0) = 1, \quad g(k) = \sum_{j=0}^k \frac{e^{\theta j}}{(k-j)!} \left( -\frac{j}{\beta} \right)^{k-j}, \quad k \geq 1. \]

This proves that there is a unique continuous solution. Notice that from equality (3.17) it follows that if \( g_\lambda \) is non-negative then it is strictly positive.

Observe that \( \beta g_\lambda'(x) = (1 - \lambda) g_\lambda(x) - g_\lambda(x - 1)1_{\{x \geq 1\}} \), and therefore it easily follows for \( \lambda \leq 0 \) that \( g_\lambda \) is an increasing function. Then \( g_\lambda \) is positive and non-integrable for \( \lambda \leq 0 \).

Let us prove that \( g_{\lambda_c} \geq 0 \). By continuity it suffices to show that \( g_{\lambda_c} \geq 0 \) on \( \mathbb{R}_+ \setminus \mathbb{N} \). Now, this follows because from relations (3.16) and (2.2) we find

\[ g_{\lambda_c}(x) = \lambda_c^{-1}(\beta - e^{(\lambda_c/\beta)(x-[x])})\mu_{x-[x]}(x) \quad \text{for } x \in \mathbb{R}_+ \setminus \mathbb{N}. \]  

(3.19)

Therefore \( g_{\lambda_c} > 0 \).
In order to prove that \( g_x \) is positive for \( 0 < \lambda < \lambda_c \), consider \( W = g_x / g_{\lambda_c} \). Assume that there exists \( x > 0 \) such that \( W'(x) = 0 \). Define

\[
x^* = \inf \{ x > 0 : W'(x) = 0 \}.
\]

Since \( W(x) = e^{(\lambda_c - \lambda)/\beta} x \) for \( x \in [0, 1] \), the function \( W \) is increasing on \([0, 1]\). Hence \( x^* \geq 1 \) and \( W \) is increasing on \([0, x^*] \). Observe that for \( x \geq 1 \)

\[
W'(x) = \frac{\lambda_c - \lambda}{\beta} W(x) + \frac{g_{\lambda_c}(x - 1)}{\beta g_{\lambda_c}(x)} (W(x) - W(x - 1)).
\]

Hence

\[
0 = \frac{\lambda_c - \lambda}{\beta} W(x^*) + \frac{g_{\lambda_c}(x^* - 1)}{\beta g_{\lambda_c}(x^*)} (W(x^*) - W(x^* - 1)),
\]

which implies that \( W(x^*) < W(x^* - 1) \). This contradicts the fact that \( W \) is increasing in \([0, x^*]\). Then \( W'(x) > 0 \) for all \( x > 0 \) which implies \( W(x) > 0 \) for \( x > 0 \). We conclude that \( g_x(x) > 0 \) for \( x > 0 \).

Notice that the above proof also shows that for \( 0 < \lambda < \lambda' \leq \lambda_c \) the function \( W(x) = g_x(x)/g_{\lambda'}(x) \) is a positive increasing function and verifies the stronger inequality

\[
W'(x) \geq \frac{\lambda' - \lambda}{\beta} W(x),
\]

and therefore \( W(x) \geq e^{(\lambda' - \lambda)/\beta} x \). This implies (3.18).

Let us show that \( g_x \) is an integrable function for \( \lambda > 0 \). From the proof below it will also follow that \( g_x \) is bounded and \( g_x(\infty) = 0 \). First notice that (3.16) implies that

\[
\int |g_x(x)| \, dx = \int_0^1 e^{\theta y} \sum_{k \geq 0} \beta^{-k} |a_k(y)| \, dy \quad \text{with} \quad a_k(y) = \sum_{j=0}^{k} (\beta e^\theta)^j \frac{(-y - j)^{k-j}}{(k-j)!}.
\]

Then, our result will be completed once we prove that the function \( \varphi \) defined by

\[
\varphi(y) = \sum_{k \geq 0} \beta^{-k} |a_k(y)|,
\]

is bounded for \( y \in [0, 1] \). We introduce the function \( \psi_y(\cdot) \) given by

\[
\psi_y(z) = \sum_{k \geq 0} z^k a_k(y).
\]

It directly follows that \( \psi_y(z) \) exists and it is analytic when \( |z| \) is small enough. In fact the equality

\[
\sum_{k \geq 0} |z|^k \sum_{j=0}^{k} (\beta e^\theta)^j \frac{|y + j|^{k-j}}{(k-j)!} = \frac{e^{\theta y}}{1 - \beta e^\theta |z| e^{-\theta}}
\]

implies \( \psi_y(z) \) is finite for \( |z| < r \) with \( r > 0 \), verifying \( \beta e^\theta r e^\theta = 1 \). From the absolute convergence of the series we also get

\[
\psi_y(z) = \frac{e^{-\theta y}}{1 - \beta e^\theta z e^{-\theta}} \quad \text{for} \quad |z| < r.
\]
The function $\psi_y(\cdot)$ is analytic in the domain which excludes the zeros of the function $\xi(z) = 1 - \beta e^{\theta} z e^{-\bar{z}}$. Now observe that $\xi(z^*) = 0$ is equivalent to $\beta e^{\theta} \rho^* \exp(-\rho^* \cos \alpha^*) = 1$ and $\alpha^* - \rho^* \sin \alpha^* = 2k\pi$ for some $k \in \mathbb{Z}$, where we have put $z^* = \rho^* e^{i\alpha^*}$. Then $\alpha^* - 2k\pi = \rho^* \sin \alpha^* = -\rho^* (\alpha^* - 2k\pi)$. We deduce that $\rho^* < 1$ implies $\alpha = 2k\pi$ and hence $\beta e^{\theta} \rho^* e^{-\beta} = 1$. Let us show that $\rho^* > \beta^{-1}$. The function $\delta(u) = \beta e^{\theta} u e^{-u}$ is increasing for $u \in [0, 1]$ and $\delta(\beta^{-1}) = e^{-\lambda/\beta} < 1$, where the strict inequality follows because $\lambda > 0$. These facts imply $\rho^* > \beta^{-1}$.

We define $\rho_0 = \inf(\rho^*, 1)$, which by previous analysis verifies $\rho_0 > \beta^{-1}$. Since $\xi(\cdot)$ does not vanish in the disk $D_0(\rho_0) = \{ |z| < \rho_0 \}$, the function $\psi_y(\cdot)$ is analytic in $D_0(\rho_0)$. Let us pick $\rho \in (\beta^{-1}, \rho_0)$. Cauchy’s inequality applied to (3.21) gives

$$|a_k(y)| \leq \rho^{-k} \max_{|z|=\rho} |\psi_y(z)|.$$

On the other hand

$$\max_{|z|=\rho} |\psi_y(z)| \leq \frac{M}{\max_{|z|=\rho} (1 - \beta e^{\theta} z e^{-\bar{z}})} \leq e^{\theta} \max_{|z|=\rho} \frac{1}{1 - \beta e^{\theta} z e^{-\bar{z}}} = M.$$

Let us come back to $\varphi(y)$ as given by (3.20). We have

$$|\varphi(y)| \leq \sum_{k \geq 0} \frac{M}{\rho^k} \beta^k = M \sum_{k \geq 0} (\beta \rho)^{-k} < \infty.$$

We conclude that $\varphi(y)$ is bounded then the integrability of $g_\lambda$ for $\lambda > 0$ follows. A similar argument shows that $g_\lambda(\infty) = 0$.

Now observe that, for $x \geq 1$,

$$\beta(g_\lambda(x) - 1) = \beta \int_0^x g_\lambda'(u) \, du = (1 - \lambda) \int_0^x g_\lambda(u) \, du - \int_0^{x-1} g_\lambda(u) \, du.$$

Passing to the limit as $x$ goes to $\infty$ we conclude that $\lim_{x \to \infty} g_\lambda(u) \, du = \beta/\lambda$. Observe that for any $\lambda > 0$ such that $g_\lambda$ is positive, the probability measure $\mu^\lambda$, with density proportional to $g_\lambda$, is conditional invariant by Theorem 3.1. Hence, $g_\lambda$ induces a conditional invariant distribution for any $\lambda \in (0, \lambda_c]$.

Finally, we must prove that $g_\lambda$ changes sign for $\lambda > \lambda_c$. If $g_\lambda$ is a positive function, then it induces a conditional invariant distribution $\mu^\lambda$, for which $P_{\mu^\lambda}(T_0 > t) = e^{-\lambda t}$ for every $t \geq 0$. Take $x > 0$ such that $\mu^\lambda((x, \infty)) \geq \delta > 0$, then we obtain $P_x(T_0 > t) \leq \delta^{-1} P_{\mu^\lambda}(T_0 > t)$, which contradicts (2.6) because $\lambda > \lambda_c$. Therefore $g_\lambda$ must change sign.

The extremal distribution $\mu^{\lambda_c}$ is called the minimal conditional invariant distribution because the distribution of the exit time has the maximal decay rate $\lambda_c$. Now we describe the relation between $\mu^{\lambda_c}$ and the limit conditional distributions ($\mu_{\alpha}$ : $\alpha \geq 0$).

**Theorem 3.3.** Let $\alpha = \lambda_c^{-1}(\beta - e^{1-(1/\beta)})$, $\eta = \lambda_c \beta^{-1}$. We have

$$\mu^{\lambda_c}(A) = \int_0^1 \alpha^{-1} e^{-\eta a} \mu_{\alpha}(A) \, da = \int_0^\infty \eta e^{-\eta a} \mu_{\alpha}(A) \, da, \quad \text{for any Borel subset } A \subseteq \mathbb{R}_+^*.$$
Proof. Let us show that the second equality is derived from the first. We have \( \mu_a = \mu_{a'} \) if \( a \equiv a' \pmod{1} \), so that \( \int_k^{k+1} e^{-\eta(a-k)} \mu_a(A) \, da = \int_0^1 e^{-\eta} \mu_a(A) \, da \). Then

\[
\int_0^\infty e^{-\eta} \mu_a(A) \, da = \sum_{k=0}^{\infty} e^{-\eta k} \int_k^{k+1} e^{-\eta(a-k)} \mu_a(A) \, da = \alpha(1 - e^{-\eta})^{-1} \mu_{\lambda_c}(A).
\]

Since \( \alpha(1 - e^{-\eta})^{-1} = \eta^{-1} \) we have proved the assertion.

Let us prove the first equality. From (3.19)
\[
g_{\lambda_c}(x) = \alpha e^{-\eta} \mu_a(x) \quad \text{for} \quad x = k + a, \ k \in \mathbb{N}, \ 0 < a < 1.
\]

On the other hand, for \( m \in \mathbb{N} \) and \( 0 \leq \varepsilon \leq 1 \) we have
\[
\mu_a([m, m + \varepsilon]) = \begin{cases} 
\mu_a(m + a) & \text{if } 0 < a < \varepsilon \\
0 & \text{if } \varepsilon \leq a \leq 1.
\end{cases}
\]

Hence
\[
\int_m^{m+\varepsilon} g_{\lambda_c}(x) \, dx = \int_0^\varepsilon \alpha e^{-\eta} \mu_a(m + a) \, da = \int_0^\varepsilon \alpha e^{-\eta} \mu_a([m, m + \varepsilon]) \, da \\
+ \int_\varepsilon^1 \alpha e^{-\eta} \mu_a([m, m + \varepsilon]) \, da.
\]

The equality also holds for all intervals, then the result follows.

4. Limit conditional process

We shall prove that the following limit distribution exists
\[
\tilde{P}_x[A] = \lim_{t \to \infty} P_x\{X_s \in A \mid T_0 > t\}, \quad (4.1)
\]
for all \( A \in \mathcal{F}_s \), \( (\mathcal{F}_s) \) being the natural filtration associated with \( (X_s) \), and any fixed \( s > 0 \).

We point out that a similar weak limit distribution has been studied for time continuous Markov processes; for the Brownian motion with constant negative drift in [9], for general one-dimensional diffusions in [2] and for Markov chains in [4].

Observe that if the limit distribution exists, then the finite dimensional distribution at times \( 0 < s_1 < \ldots < s_\ell \) is concentrated at the points \( x_1, \ldots, x_\ell > 0 \), verifying \( k_i = x_i - x + \beta s_i \in \mathbb{N} \) and such that \( k_i \) is non-decreasing with \( i \). We have
\[
P_x\{X_{s_1} = x_1, \ldots, X_{s_\ell} = x_\ell \mid T_0 > t\}
= \mathbb{E}_x \left( I_{(X_{s_1} = x_1)} \ldots I_{(X_{s_\ell} = x_\ell)} I_{(T_0 > s_\ell)} P_{s_\ell} \{T_0 > t - s_\ell\} \right). \]

Then if we are able to show that the following limit exists; i.e.
\[
\Pi_x(s, x) = \lim_{t \to \infty} \frac{P_x\{T_0 > t - s\}}{P_x\{T_0 > t\}},
\]
for any \( x > 0 \) in the form \( x = x + \beta s \in \mathbb{N} \), the limit distribution \( \tilde{P}_x \) will exist and it will correspond to the distribution of a Markov process \( (Z_t, t \geq 0) \), with transitions given by
\[
\tilde{P}_x\{Z_s = x_s\} = \Pi_x(s, x) P_x\{X_s = x_s, T > s\}.
\]
**Lemma 4.1.** For $s > 0, x_s > 0$ such that $x_s - x + \beta s \in \mathbb{N}$, we have

$$
\lim_{t \to \infty} \frac{P_x(T_0 > t - s)}{P_x(T_0 > t)} = \frac{x_s e^{s - x} \log \beta}{x}.
$$

**Proof.** We have $t = t_n^{a,x} = (x + n + a)/\beta$ for some $a \in \{0, 1\}, n \in \mathbb{N}$. We can assume $n$ is large and we explicit in $t_n^{a,x}$ its dependence on $x$. Denote $k = x_s - x + \beta s$ and $x' = x_s$. We have $t - s = (x' + n - k + a)/\beta = t_n^{a,x'}$. Hence

$$
\frac{P_{x'}(T_0 > t - s)}{P_x(T_0 > t)} = \frac{P_{x'}(T_0 > t_n^{a,x'})}{P_x(T_0 > t_n^{a,x})} = \frac{P_{x'}(T_0 > t_n^{0,x'})}{P_x(T_0 > t_n^{0,x})}.
$$

From (2.3) we find that

$$
\lim_{t \to \infty} \frac{P_{x'}(T_0 > t - s)}{P_x(T_0 > t)} = \lim_{n \to \infty} x' e^{x' + n - k - t_n^{0,x'}} \frac{\beta^n}{x e^{x + n - t_n^{0,x}} \beta^n}.
$$

Now we use $k = x' - x + \beta s$, $t_n^{0,x} - t_n^{0,x'} = s$. Then

$$
\lim_{t \to \infty} \frac{P_{x'}(T_0 > t - s)}{P_x(T_0 > t)} = \frac{x'}{x} e^{s - x} \frac{\log \beta}{x}.
$$

We now characterize the limit distribution of the process.

**Theorem 4.2.** The limit distribution $\bar{P}_x$ given in (4.1) exists and it corresponds to the law of the Markov process

$$
Z_t = x + M_t - \beta t,
$$

where $M_t$ is a counting process in $\mathbb{R}_+$ with predictable intensity $\beta((M_{s-} + 1)/M_{s-})1_{\{M_{s-} > 0\}}$.

**Proof.** Denote by $\bar{E}_x$ the mean expected value operator associated with $\bar{P}_x$. From the last lemma we have

$$
\bar{E}_x(f(Z_s)) = (x e^{x \log \beta})^{-1} e^{x - s} \bar{E}_x(f(X_s)X_s e^{X_s \log \beta}, T_0 > s).
$$

Then the generator of the process with law $\bar{P}_x$ is

$$
\bar{L} f(x) = \lambda_c f(x) + (x e^{x \log \beta})^{-1} (\bar{L} g)(x),
$$

with $g(x) = f(x) e^{x \log \beta}$. Then

$$
\bar{L} f(x) = \lambda_c f(x) - \beta f'(x) - \left(\frac{\beta}{x} + \beta \log \beta\right) f(x) + \beta f(x + 1) \frac{x + 1}{x} - f(x)
$$

$$
= -\beta f'(x) + (f(x + 1) - f(x)) \beta \left(\frac{x + 1}{x}\right).
$$

From this fact, we easily obtain that

$$
M_t = \int_0^t \beta \frac{M_{s-} + 1}{M_{s-}} 1_{\{M_{s-} > 0\}} ds
$$

is a martingale, which completes the proof.
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