A BRANCHING PROCESS WITH DISASTERS

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Abstract

A population process is considered where particles reproduce according to an age-dependent branching process, and are subjected to disasters which occur at the epochs of an independent renewal process. Each particle alive at the time of a disaster, survives it with probability \( p \) and the survival of any particle is assumed independent of the survival of any other particle. The asymptotic behavior of the mean of the process is determined and as a consequence, necessary and sufficient conditions are given for extinction.

BRANCHING MODEL; GENERALIZED BRANCHING PROCESS; RENEWAL THEORY

1. Introduction

We consider a one-dimensional population process where particles reproduce according to an age-dependent branching process, and are subjected to 'disasters' which occur at the random times \( (\tau_i) \). The \( (\tau_i) \) are assumed to be the epochs of a renewal process which proceeds independently of the population. When a disaster happens each particle has probability \( p \) \((0 < p < 1)\) of surviving, and the survival of any particle is independent of the survival of all other particles.

Let \( Z(t) \) be the number of particles in the population alive at time \( t \) \((Z(0) = 1)\). It follows that the number of particles surviving the \( i \)th disaster is a binomial random variable with parameters \( p \) and \( Z(\tau_{i−}) \), where \( Z(\tau_{i−}) \) is the number of particles alive just before the \( i \)th disaster.

The above process can be applied to many physical phenomena providing the appropriate interpretation is given for the disaster. For example, disasters could be epidemics, famine, war or earthquakes. In general the survival mechanism is much more complicated than the one used here, but still it is of interest to consider our simplified model. As an example of a process where the survival mechanism does resemble the one assumed, we consider a bacterial population that is randomly treated with radiation. It is not unreasonable to assume that each particle survives the radiation treatment with probability \( p \).

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The first goal in our study of the \( \{Z(t)\} \) process is to determine the a.e. behavior of \( E[Z(t) \mid F] \) where \( F = \sigma \)-field generated by the \( \{T_i = \tau_i - \tau_{i-1}\}_{i \geq 1} \). It is shown in Section 2 that there exists a constant \( \gamma \) such that \( \gamma \leq 0 \) implies \( \lim_{t \rightarrow \infty} E[Z(t) \mid F] = 0 \) a.e. and \( \gamma > 0 \) implies \( \lim_{t \rightarrow \infty} E[Z(t) \mid F] = \infty \) a.e. The value \( \gamma \) is the 'Malthusian parameter' of the population. The unconditional expectation \( E[Z(t)] \) is also examined and it is proven that there exists a constant \( \gamma_1 \) such that \( E[Z(t)] \sim e^{\gamma_1 t} \). It turns out that \( \gamma < \gamma_1 \).

In Section 3, we deal with the question of extinction. The a.e. behavior of \( E[Z(t) \mid F] \) is the determining factor for this problem. In particular, we prove that extinction is a sure event if \( \gamma \leq 0 \). When \( \gamma > 0 \), all nonzero states are proven transient.

In Section 4, a special case is studied which corresponds to when the natural lifetime of a particle has a negative exponential distribution. The natural lifetime of a particle is the length of its life assuming that it survives all disasters that occur while it is alive. In this situation, the number of particles alive at time \( t \), properly normalized is a continuous time martingale when conditioned on \( F \). Using this fact we are able to determine the asymptotic behavior of the process when explosion occurs.

At this point we introduce some notation and assumptions. We assume that each particle has a natural lifetime of length \( L \), and we put \( G(t) = P[L \leq t] \). When natural death occurs, the random number of particles \( M \) born has probability generating function (p.g.f.) \( f(s) \). Let \( X(t) \) denote the number of particles alive at time \( t \) disregarding the disasters. It follows from the assumptions in the beginning of the section that the \( \{X(t)\} \) process is an age-dependent branching process having lifetime distribution \( G \) and offspring p.g.f. \( f(s) \). The reader should also note that the \( \{X(t)\} \) process is independent of the \( \{\tau_i\} \) process and \( Z(t) \leq X(t) \) w.p.1 for all \( t \geq 0 \).

The disasters are assumed to arrive at the epochs \( (\tau_i) \) of a renewal process. Thus, \( \tau_i = \sum_{j=1}^{i} T_j, \; i \geq 1, \; (\tau_0 = 0) \), where the \( \{T_i\} \) are independent and identically distributed with common distribution function \( F \).

For the remainder of this paper we make the following assumptions:

(i) \( G(0+) = F(0+) = 0; \; G \) and \( F \) are nonlattice.

(ii) \( 0 \leq P[M = 0] + P[M = 1] < 1; \; \frac{\partial f(s)}{\partial s} \bigg|_{s=1}^- = m < \infty \).

(iii) \( E[T_1] = \lambda < \infty \).

(iv) \( P[Z(0) = 1] = 1 \).

2. The mean

The first step in determining the behavior of the mean is to define the renewal function, \( N(\cdot) \), associated with the \( \{\tau_i\} \) process. Put

\[ N(t) = k \text{ if } \tau_k \leq t < \tau_{k+1}; \; k \geq 0. \]
We then have the following theorem.

**Theorem 2.1.**

\[ E\{Z(t) \mid F\} = p^{N(t)} E\{X(t)\}; \quad t \geq 0. \]

**Proof.** $Z(t)$ can be written as

\[ Z(t) = \sum_{i=1}^{X(t)} \eta_i; \quad t \geq 0 \]

where,

\[ \eta_i = \begin{cases} 1, & \text{if the ancestry of the } i\text{th particle alive at} \\
 & \text{time } t \text{ in the } X \text{ process survived all } N(t) \\
 & \text{disasters up to time } t. \\
0, & \text{otherwise.} \end{cases} \]

Since the $\{\eta_i\}_{i=1}^{X(t)}$ depend only on the $\{N(t)\}$ process, which is assumed independent of the $\{X(t)\}$ process, we conclude that,

\[ E\{Z(t) \mid F\} = p^{N(t)} E\{X(t)\}. \]

As an immediate corollary, we obtain the following.

**Corollary 2.1.**

\[ E\{Z(t)\} = E\{p^{N(t)}\} E\{X(t)\}, \quad t \geq 0. \]

We now determine the asymptotic behavior of the above quantities.

**Theorem 2.2.** Assume that there exists a constant $\alpha$ such that,

\[ m \int_0^\infty e^{-\alpha t} dG(t) = 1. \]  

Define, $\gamma = \lambda^{-1} \log p + \alpha$. Then,

\[ \gamma \leq 0 \Rightarrow \liminf_{t \to \infty} E\{Z(t) \mid F\} = 0 \quad \text{a.e.} \]

\[ \gamma > 0 \Rightarrow \liminf_{t \to \infty} E\{Z(t) \mid F\} = \infty \quad \text{a.e.} \]

Suppose, in addition there exists a constant $\beta$ such that

\[ p \int_0^\infty e^{\beta t} dF(t) = 1. \]

Then,
\begin{align*}
(2.4) 
E[Z(t)] & \sim e^{\gamma t}, \quad \gamma_1 = \alpha - \beta.
\end{align*}

Proof. It is well known ([6], p. 143) that \(E[X(t)]\) is asymptotic to \(e^{\alpha t}\), providing (2.1) is satisfied. If \(m \geq 1\), then \(\alpha\) always exists and if \(m < 1\), \(\alpha\) may not exist ([3], Chapter 4).

Since \(N(t) \sim \lambda^{-1} t\) a.e. ([5], Chapter 11), (2.2) is clear when \(\gamma \neq 0\). If \(\gamma = 0\), it suffices to show that

\begin{align*}
(2.5) 
\liminf_{t \to \infty} E[Z(\tau) | F] &= 0 \text{ a.e.}
\end{align*}

Observe that

\begin{align*}
E[Z(\tau) | F] &= p^i E[X(\tau_i)] \\
&\leq K p^i \exp \left\{ \alpha \sum_{j=1}^{i} T_j \right\}
\end{align*}

where \(K\) is some constant. Since \(\gamma = 0\),

\begin{align*}
p^i \exp \left\{ \alpha \sum_{j=1}^{i} T_j \right\} &= \exp \left\{ \sum_{j=1}^{i} W_j \right\}
\end{align*}

where the \(\{W_j\}_{j=1}^{i}\) are i.i.d. with \(E[W_1] = 0\). (2.5) now follows because \(\liminf_{t \to \infty} \left[ \sum_{j=1}^{i} W_j \right] = -\infty\) a.e.

To prove (2.4) we need the behavior of \(B_p(t) = E[p^{N(t)}]\). This can easily be obtained from the basic Renewal Theorem ([5], p. 349). It is not hard to demonstrate that \(B_p(t)\) satisfies the renewal equation

\begin{align*}
B_p(t) = 1 - F(t) + p \int_{0}^{t} B_p(t-y) dF(y), \quad t \geq 0.
\end{align*}

Standard renewal type arguments can be applied to show that \(B_p(t)\) is asymptotic to \(e^{-\beta t}\) \((\beta > 0)\), providing there exists a \(\beta\) satisfying (2.3). (2.4) now follows.

Even if either \(\alpha\) or \(\beta\) does not exist, some remarks about the behavior of \(E[Z(t)]\) can be made. For definiteness assume that \(\alpha\) exists and \(\beta\) does not. There is a large class of distributions for which the asymptotic behavior of \(B_p(t)\) can still be given. We make the following definition.

**Definition.** The 'sub exponential class' \(\mathcal{S}\) consists of all distribution functions \(G\) such that

\begin{align*}
\lim_{t \to \infty} \frac{1 - G^{*2}(t)}{1 - G(t)} = 2
\end{align*}

where
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\[ G^{*2}(t) = \int_0^t G(t-u) dG(u). \]

For a discussion of this class of distribution functions the reader is referred to [3], Chapter 4. Their results show that if \( F \) is a member of \( \mathcal{S} \), then

\[ B_p(t) \sim 1 - F(t). \]

Hence

\[ E\{Z(t)\} \sim e^{\alpha t}[1 - F(t)]. \]

In particular, it is proven in [3] that if \( \alpha > 0 \), \( e^{\alpha t}[1 - F(t)] \to \infty \) as \( t \to \infty \). Thus \( \alpha > 0 \) implies that \( E\{Z(t)\} \to \infty \) as \( t \to \infty \).

In order to avoid technical difficulties we will assume for the remainder of this paper that both \( \alpha \) and \( \beta \) exist.

3. Extinction

In this section we deal with the question of extinction. As already indicated in the introduction, we need to consider the behavior of \( E\{Z(t) \mid F\} \). The reason for this is that in the unconditional \( \{Z(t)\} \) process, individual particles do not have independent lines of descent. They are correlated, since the times of disasters are the same for all lines of descent. However, if we condition on the times of the disasters, then indeed we do obtain independent lines of descent. This behavior is markedly similar to branching processes with random environments ([1], [2]).

We now state and prove our main result. Without loss of generality we can assume \( m > 1 \), for since \( Z(t) \leq X(t) \) a.e., for all \( t \geq 0 \), when \( m \leq 1 \),

\[ 0 \leq \limsup_{t \to \infty} Z(t) \leq \limsup_{t \to \infty} X(t) = 0, \text{ a.e.} \]

In this case \( \alpha > 0 \).

**Theorem 3.1.** w.p.1,

\[ (3.1) \quad P\{ \lim_{t \to \infty} Z(t) = 0 \mid F\} = 1 \quad \text{iff} \quad \gamma \leq 0. \]

**Proof.** Suppose first that \( \gamma \leq 0 \). By Fatou’s Lemma and Theorem 2.2,

\[ 0 \leq E\{\liminf_{t \to \infty} Z(t) \mid F\} \leq \liminf_{t \to \infty} E\{Z(t) \mid F\} = 0 \text{ a.e.} \]

Hence, \( P\{\liminf_{t \to \infty} Z(t) = 0 \mid F\} = 1 \) a.e. However, since \( Z(t) \) is integer valued and 0 is an absorbing state,

\[ \liminf_{t \to \infty} Z(t) = 0 \Rightarrow \lim_{t \to \infty} Z(t) = 0. \]

(3.1) now follows.

We now consider the case \( \gamma > 0 \). To handle this we use the technique of Karlin
and Kaplan [7]. Consider the \{Z(t)\} process conditioned on \(F\), and let \(Y_n(a)\) denote the vector of the ages of the respective particles that are alive right after the \(n\)th disaster \((n \geq 1)\), given that the initial particle was of age \(a\), \(a \geq 0\). (If \(a = 0\), we put \(Y_n = Y_n(0)\).) The \(\{Y_n(a)\}\) process is Markov and a 'general branching process' as defined by Harris ([6], Chapter 3). However, one should note that it is not time homogeneous.

Fix \(A > 0\), to be determined later. For any vector \(x\), we set \(\eta_A(x)\) equal to the number of components of \(x\) less than or equal to \(A\), and define the auxiliary process:

\[
V_j(a) = \eta_A(Y_j(a)), \quad j \geq 1
\]

\[
= \text{number of particles alive after the } j\text{th disaster of age } \leq A, \text{ given the initial particle was of age } a.
\]

(If \(a = 0\), we write \(V_j\) for \(V_j(0)\).)

Fix \(D > 0\), to be determined later. Since \(\gamma > 0\), we can always find an integer \(k_0\) so that

\[
(3.2) \quad \beta = \log D + k_0\gamma > 0.
\]

Define the random variables

\[
\rho_i = D^i p^{ik_0} \exp \left\{ \alpha \sum_{j=1}^{ik_0} T_j \right\}, \quad i \geq 1.
\]

Let \(0 < \varepsilon < \beta\). It follows from the Strong Law of Large Numbers that there exists a finite, integer valued random variable \(I\), such that w.p.1,

\[
(3.3) \quad \rho_i \geq e^{(\beta - \varepsilon)i}, \quad i \geq I.
\]

Since we are conditioning on \(F\) we can treat the \(\{\rho_i\}\) as a sequence of constants satisfying (3.3).

To prove extinction is not a sure event, it is sufficient to show that

\[
(3.4) \quad \lim_{n \to \infty} P\{V_{ik_0} > \rho_i; 1 \leq i \leq n \mid F\} > 0 \quad \text{a.e.}
\]

or equivalently,

\[
(3.5) \quad \lim_{n \to \infty} \prod_{j=1}^{n} P\{V_{jk_0} \geq \rho_j \mid V_{ik_0} \geq \rho_i; 1 \leq i \leq j-1; F\} > 0 \quad \text{a.e.}
\]

Our problem reduces to estimating for each \(j \geq 1\),

\[
\Theta_j = P\{V_{jk_0} > \rho_j \mid V_{ik_0} > \rho_i; 1 \leq i \leq j-1, F\}.
\]

To make this estimate we need some simple facts about age-dependent branching processes. Without loss of generality we can assume that \(d^2 f/ds^2(1-) < \infty\). A
simple truncation argument shows this. For an age-dependent branching process we define

\[ X_a(t, A) = \text{number of particles alive at time } t \text{ of age } \leq A \text{ given the original particle was of age } a. \]

(If \( A = \infty \), we just write \( X_a(t) \).)

Choose \( A \) so that \( 1 - G(A) > 0 \). In this case it is not hard to show that there exist positive constants \( B(A) \) and \( C(A) \) such that

\[
\inf_{t > 0} \frac{\text{E} \{ X_a(t, A) \} e^{-at}}{a < A} \geq B(A)
\]

and

\[
\sup_{t > 0} \frac{\text{E} \{ X_a^2(t, A) \} e^{-2at}}{a < A} \leq C(A).
\]

Let \( \tilde{y} \) be any positive finite dimensional vector such that \( \eta_a(\tilde{y}) \geq [\rho_{j-1}]\lfloor x \rfloor \) equals the greatest integer in \( x \), and let \( \{ a_i \} \) be the magnitudes of those components of \( \tilde{y} \) less than or equal to \( A \). Consider:

\[
T_{j}(\tilde{y}) = P \{ V_{\tau_{k0}} > \rho_j | V_{\tau_{i0}} > \rho_i, 1 \leq i \leq j-1; Y_{(j-1)\tau_{k0}} = \tilde{y}; F \}.
\]

It follows from Section 1, that

\[
\Theta_{j}(\tilde{y}) \geq P \left\{ \sum_{i=1}^{[\rho_{j-1}]} \delta_i > \rho_{j-1} \right\}
\]

where the \( \{ \delta_i \} \) are independent random variables and \( \delta_i \) has the same distribution as \( V_{\tau_{k0}}(a_i) \) except that the time of the \( k_0 \)th disaster is \( \tau_{j\tau_{k0}} - \tau_{(j-1)\tau_{k0}} \) instead of \( \tau_{\tau_{k0}} \). Using (3.6) and (3.7) it is not difficult to show that

\[
E\{ \delta_i \} \geq B(A) \rho_{j-1} \exp(\alpha(\tau_{j\tau_{k0}} - \tau_{(j-1)\tau_{k0}}))
\]

and

\[
\text{Var}(\delta_i) \leq C(A) \exp(2\alpha(\tau_{j\tau_{k0}} - \tau_{(j-1)\tau_{k0}})) = \zeta(j).
\]

Let \( \varepsilon > 0 \), set \( D_j = B(A)(1 - \varepsilon) \), and choose \( k_0 \) so that (3.2) is satisfied. Then for \( j \) sufficiently large,

\[
P \left\{ \sum_{i=1}^{[\rho_{j-1}]} \delta_i > \rho_j \right\} \geq P \left\{ \sum_{i=1}^{[\rho_{j-1}]} (\delta_i - E\{ \delta_i \}) > \rho_j(1 - \varepsilon)^{-1} \varepsilon \right\}
\]

\[
\geq 1 - \frac{\text{Var}(\delta_{j})}{(\rho_j(1 - \varepsilon)^{-1} \varepsilon)^2}
\]
\[ \geq 1 - \frac{[\rho_{j-1}] \zeta(j)}{(\rho_j (1-\varepsilon)^{-1}\varepsilon)^2}, \]

the last two inequalities coming from Chebychev’s Inequality and (3.9). Since the final lower bound does not depend on \( \bar{y} \), we obtain:

\[ \Theta_j \geq 1 - \frac{[\rho_{j-1}] \zeta(j)}{(\rho_j (1-\varepsilon)^{-1}\varepsilon)^2}. \]

Thus (3.5) will follow, providing we can show

\[ Q = \sum_{j=1}^{\infty} \frac{[\rho_{j-1}] \zeta(j)}{\rho_j^2} < \infty \text{ a.e.} \]

Using the definition of \( \rho_j \) and (3.3) it is not difficult to see that w.p.1,

\[ \sum_{j=1}^{\infty} \exp\{\alpha(\tau_{jk} - \tau_{(j-1)k})\} e^{-j(\beta-\varepsilon)/2} < \infty \Rightarrow Q < \infty. \]

But,

\[ \sum_{j=1}^{\infty} P\{\exp(\alpha(\tau_{jk} - \tau_{(j-1)k})) > e^{j(\beta-\varepsilon)/2} \} = \sum_{j=1}^{\infty} P\{\alpha \tau_{k0} > \left( \frac{\beta - \varepsilon}{2} \right) j \} < \infty, \]

since \( E(\tau_{k0}) < \infty \). Hence by the Borel Cantelli Lemma, we see that the sum on the left in (3.10) is indeed finite a.e. This completes the proof of the theorem.

We now consider the behavior of the process when extinction is not a sure event. In particular, we will show that all the nonzero states are transient.

**Theorem 3.2.**

\[ P\{\lim_{t \to \infty} Z(t) = 0\} + P\{\lim_{t \to \infty} Z(t) = \infty\} = 1. \]

The proof of the above result is based on the following simple lemma.

**Lemma 3.1.** Let \( \{Y(t)\}_{t \geq 0} \) be a positive integer valued process with 0 as an absorbing state. Let \( D = \{\lim_{t \to \infty} Y(t) = 0\} \) and \( F_t = \sigma(Y(s), 0 \leq s \leq t). \) (If \( \mathcal{S} \) is any collection of random variables, \( \sigma(\mathcal{S}) \) is the smallest \( \sigma \)-field generated by \( \mathcal{S} \).) Suppose that

\[ \limsup_{t \to \infty} E\{I_D | F_t\} > 0 \text{ whenever } \liminf_{t \to \infty} Y(t) < \infty. \]

\( (I_D \text{ is the indicator function of the set } D). \)
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Then

\[ P\{\lim_{t \to \infty} Y(t) = 0\} + P\{\lim_{t \to \infty} Y(t) = \infty\} = 1. \]

**Proof.** By a suitable martingale theorem ([4], p. 93)

(3.12)

\[ \lim_{t \to \infty} E\{I_D \mid F_t\} = I_D \quad \text{a.e.} \]

Consider any sample path of the \( Y(t) \) process not in \( D \) and assume \( \liminf_{t \to \infty} Y(t) \)

\(< \infty \). It follows by our assumption that this implies \( \limsup_{t \to \infty} E\{I_D \mid F_t\} > 0 \)

which in turn from (3.11) implies that the sample path is in \( D \). This is a contradiction.

**Proof of Theorem 3.2.** There are two cases to consider depending on whether \( f(0) > 0 \) or \( f(0) = 0 \). Consider the first case. We will show that (3.11) holds. Define

\[ q = P\{\lim_{t \to \infty} X(t) = 0 \mid X(0) = 1\}. \]

Since \( f(0) > 0 \), \( q > 0 \). Using the assumptions of Section 1, it is not difficult to see that,

\[ E\{I_D \mid F_t\} \geq q^k \]

whenever \( Z(t) = k \).

This implies (3.11).

When \( f(0) = 0 \) the above argument does not work since \( q = 0 \). However, the result is still true. To prove this we need the \( \{Y_n\} \) process introduced earlier in this section. For any vector \( \vec{y} \), put \( |\vec{y}| \) equal to the number of its components. Let

\[ D = \{\lim_{n \to \infty} |Y_n| = 0\} \]

\[ F_n = \sigma(Y_1, Y_2, \ldots, Y_n, T_1, \ldots, T_n). \]

In order to invoke Lemma 3.1, it is sufficient to prove that for any integer \( k \), there exists a positive constant \( C_k \) such that

(3.13)

\[ E\{I_D \mid F_n\} \geq C_k \text{ whenever } |Y_n| = k. \]

Let \( \{y_i\} \) be the components of \( Y_n \). Then

\[ E\{I_D \mid F_n\} = E\{E\{I_D \mid F_n, T_{n+1}\} \mid F_n\} \]

and

\[ E\{I_D \mid F_n, T_{n+1}\} \geq \prod_{j=1}^{[Y_n]} E\{(1 - p)^X_{x_j(T_n+1)}\} \]

where \( \{X_{x}(t)\} \) is an age-dependent branching process with the initial particle of age \( x \). By Jensen’s Inequality,
\[ E\{(1-p)^X_{Y(T_{n+1})}\} \geq (1-p)^{E(X_{Y(T_{n})})}. \]

For any \( x \), \( E\{X_{Y(t)}\} \leq 1 + E\{X(t)\}. \) Hence,
\[ E\{I_{D} \mid F_{n}, T_{n+1}\} \geq (1-p)^{(1+E(X_{n+1}))m}\left|Y_{n}\right|. \]

Finally, since \( T_{n+1} \) is independent of \( F_{n} \), we obtain
\[ E\{I_{D} \mid F_{n}\} \geq (1-p)^{(1+mE(X(T_{0})))}Y_{n}|F(T_{0})|. \]

(\( T_{0} \) is chosen so that \( F(T_{0}) > 0. \))

The last inequality implies (3.13). To complete the proof we note that the \( \{Z(t)\} \) process is monotonic increasing between disasters.

Remarks. (1) It necessarily follows from Theorem 3.2 that
\[ P\left(\lim_{t \to \infty} Z(t) = 0 \mid F\right) + P\left(\lim_{t \to \infty} Z(t) = \infty \mid F\right) = 1 \text{ w.p.l.} \]

This result is in the spirit of branching processes in random environments [1].

(2) If \( f(0) > 0 \), we need only assume that \( \lim_{t \to \infty} \tau_{i} = \infty \) a.e. to prove Theorem 3.2. If \( f(0) = 0 \), the renewal assumption can still be dropped, providing we assume for example that there exists a constant \( L \) such that \( P\{T_{i} \leq L\} = 1 \), \( i \leq 1 \). We conjecture that when \( f(0) = 0 \), Theorem 3.2 holds with no assumption on the \( \{\tau_{i}\} \) other than \( \lim_{t \to \infty} \tau_{i} = \infty \) a.e.

4. The exponential case

In this section we take \( G(t) = 1 - e^{\lambda - t} \); \( \lambda > 0 \).

Define the normalization random variables
\[ A(t) = E\{Z(t) \mid F\} = p^{N(t)}e^{\lambda t}; \quad t \geq 0 \]
and put \( W(t) = Z(t)/A(t) \). We then have the following theorem.

Theorem 4.1. Assume \( G(t) = 1 - e^{\lambda - t} \); \( \lambda > 0 \). Then,
\[ \lim_{t \to \infty} W(t) = W \text{ exists w.p.l.} \]
\[ (4.1) \]

If in addition \( \frac{d^2f}{ds^2}(1-) < \infty \), \( W \) is nondegenerate.

In fact,
\[ P\{W > 0 \mid F\} > 0 \text{ w.p.l.} \]

Proof. Define the family of \( \sigma \)-fields
\[ G_{t} = \sigma\{Z(s), 0 \leq s \leq t; F\}; \quad t \geq 0. \]
To prove (4.1) it suffices to show that \( \{W(t), \mathcal{G}_t\}_{t \geq 0} \) is a martingale with respect to the conditional probability \( P(\cdot | F) \). To prove the martingale property, we use Theorem 2.1 and the observation that \( E[X(t)] = \exp(\alpha t) \) for \( G \) exponential. Indeed,

\[
E\{W(t+s) \mid \mathcal{G}_t, F\} = p^{-N(t+s)}e^{-\alpha(t+s)}E\{Z(t+s) \mid \mathcal{G}_t, F\} \\
= p^{-N(t+s)}e^{-\alpha(t+s)}Z(t)p^{N(t+s)-N(t)}e^{\alpha s} \\
= W(t).
\]

Thus \( \lim_{t \to \infty} W(t) = W \) exists w.p.l. In order to prove that \( T \) is non-degenerate, it suffices to show that \( \limsup_{n \to \infty} E\{W^2(\tau_n) \mid F\} < \infty \) w.p.l.

By the martingale property, we have:

\[
E\{W^2(\tau_n) \mid F\} = E\{W^2(\tau_{n-1}) \mid F\} + E\{(W(\tau_n) - W(\tau_{n-1}))^2 \mid F\}.
\]

But

\[
E\{(W(\tau_n) - W(\tau_{n-1}))^2 \mid F\} = E\{E\{(W(\tau_n) - W(\tau_{n-1}))^2 \mid \mathcal{G}_{\tau_{n-1}}\} \mid F\} \\
= p^{-2n}e^{-2 \alpha T_n}E\{E(X(\tau_n) - pe^{\alpha T_n}Z(\tau_{n-1}))^2 \mid \mathcal{G}_{\tau_{n-1}}\} \mid F\}.
\]

Conditioned on \( \mathcal{G}_{\tau_{n-1}}, Z(\tau_n) = \sum_{j=1}^{Z(\tau_{n-1})} Y_j \) where the \( \{Y_j\}_{j=1}^{Z(\tau_{n-1})} \) are i.i.d. with \( Y_1 \) having the same distribution as the number of descendants of one ancestor surviving the first disaster which occurs at time \( T_n \). Note that \( E(Y_1) = pe^{\alpha T_n} \). Thus, \( Z(\tau_n) \) follows an \( \text{Poisson}(pe^{\alpha T_n}) \) distribution. Hence,

\[
E\{(Z(\tau_n) - pe^{\alpha T_n}Z(\tau_{n-1}))^2 \mid \mathcal{G}_{\tau_{n-1}}\} = E\left\{\left[\sum_{j=1}^{Z(\tau_{n-1})} (Y_j - E(Y_j))\right]^2\right\} \\
= Z(\tau_{n-1}) \text{Var}(Y_1) \\
\leq KZ(\tau_{n-1})e^{2 \alpha T_n}
\]

for some constant \( K \). Hence,

\[
E\{(W(\tau_n) - W(\tau_{n-1}))^2 \mid F\} = KW(\tau_{n-1})p^{-2n}e^{-2 \alpha T_{n-1}}
\]

and so

\[
\limsup_{n \to \infty} E\{W^2(\tau_n) \mid F\} \leq \sum_{i=1}^{\infty} KW(\tau_{i-1})p^{-i}e^{-\alpha T_{i-1}} \\
< \infty \quad \text{w.p.l.}
\]

The last inequality is true since \( \limsup_{t \to \infty} W(\tau_i) < \infty \) a.e. and

\[
\sum_{i=1}^{\infty} p^{-i}e^{-\alpha T_{i-1}} < \infty \quad \text{a.e.}
\]
Remark. For general $G$, $W(t)$ is not a martingale. It turns out that we can still define a relevant martingale. Define the function

$$V(x) = \int_0^\infty e^{-\alpha y} dG_x(y), \quad x \geq 0$$

where

$$G_x(y) = \frac{G(x + y) - G(x)}{1 - G(x)}, \quad y \geq 0.$$ 

At time $t$, let $x_1, x_2, \ldots, x_{Z(t)}$ denote the ages of the particles alive at time $t$. We then define

$$V_t = \sum_{i=1}^{Z(t)} V(x_i), \quad t \geq 0.$$ 

It is not difficult to show that $\{A^{-1}(t) V_{t-n}, G_i\}_{n=0}^\infty$ is a martingale with respect to $P(\cdot | F)$. This fact will be exploited in future paper [8] to prove that Theorem 4.1 holds for general $G$.

5. Generalizations

In this section we will indicate some generalizations of the model. Let us first consider the times when disasters occur. The key to the analysis is the behavior of $E[Z(t) | F]$ and hence, we would want conditions on the inter-arrival times $\{T_i\}$ which imply (2.2). When $\gamma \neq 0$, all that is required is that $\lim N(t)/t = \lambda^{-1}$ a.e. This property, for example, is satisfied by any stationary ergodic sequence. In this case, the proofs of Theorems 3.1 and 3.3 go through without change. Note that the proof of Theorem 3.2 does require the independence assumption. When $\gamma = 0$, stronger assumptions are needed. It seems that one could have a stationary ergodic sequence satisfying some kind of mixing condition. The results of Section 4 hold under the assumption that the $\{T_i\}$ form a stationary ergodic sequence without qualification.

Let us now consider the probability of survival. One might assume that $p$ depends upon the disaster. In particular, let $p_i$ be a random variable where $p_i$ is the probability a particle survives the $i$th disaster. If we assume that $\{(p_i, T_i)\}_{i=1}^\infty$ is a stationary ergodic sequence then the results indicated above all go through with the only change being $\gamma = E[\log p_1] \lambda^{-1} + \alpha$.

One could also reason that $p$ should depend on the age of the particle at the time of the disaster. We then would assume that there exists a measurable function $p(\cdot)$ taking values between 0 and 1 such that a particle of age $x$ survives a disaster with probability $p(x)$. The analysis of this process is an open question and seems difficult.
References


