THE COLLISION BRANCHING PROCESS

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Abstract

We consider a branching model, which we call the collision branching process (CBP),
that accounts for the effect of collisions, or interactions, between particles or individuals.
We establish that there is a unique CBP, and derive necessary and sufficient conditions for
it to be nonexplosive. We review results on extinction probabilities, and obtain explicit
expressions for the probability of explosion and the expected hitting times. The upwardly
skip-free case is studied in some detail.

Keywords: Regularity; extinction probability; hitting time

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1. Introduction

Consider an ensemble of particles that evolves as follows. Collisions between particles occur
at random and, whenever two particles collide, they are removed and replaced by \( k \) ‘offspring’
with probability \( p_k \) (\( k \geq 0 \)), independently of other collisions. In any small time interval
\( (t, t + \Delta t) \) there is a positive probability \( \theta \Delta t + o(\Delta t) \) that a collision occurs, and the chance
of two or more collisions occurring in that time interval is \( o(\Delta t) \).

Suppose that there are \( i \) particles present at time \( t \). Assuming that all pair interactions are
equally likely, then, after time \( \Delta t \), there will be \( j \) particles with probability \( \binom{i}{2} \theta p_{j-i+2} \Delta t + o(\Delta t) \). We may therefore take \( X(t) \), the number of particles present at time \( t \), to be a continuous-
time Markov chain with nonzero transition rates \( q_{ij} = \binom{i}{2} b_{j-i+2} \), \( j \geq i - 2 \), \( i \geq 2 \), where
\( b_2 = -\theta (1 - p_2) \) and, for \( j \neq 2 \), \( b_j = \theta p_j \).

This leads us to the following formal definition.

Definition 1. A conservative \( q \)-matrix \( Q = \{q_{ij}, i, j \in \mathbb{Z}_+\} \) is called a collision branching
\( q \)-matrix (CB \( q \)-matrix) if it takes the following form:

\[
q_{ij} = \begin{cases} 
\binom{i}{2} b_{j-i+2} & \text{if } j \geq i - 2, \ i \geq 2, \\
0 & \text{otherwise,}
\end{cases}
\]  

(1)

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where

\[ b_j \geq 0 \quad (j \neq 2) \quad \text{and} \quad -b_2 = \sum_{j \neq 2} b_j < +\infty, \tag{2} \]

together with \( b_0 > 0, b_1 > 0, \) and \( \sum_{j=3}^{\infty} b_j > 0. \)

The conditions \( b_0 > 0 \) and \( \sum_{j=3}^{\infty} b_j > 0 \) are essential, while the condition \( b_1 > 0 \) is imposed for convenience; all our conclusions hold true with some minor and obvious adjustments if this latter condition is removed.

This model has been considered by several authors, including Ezhov [9] and Kalinkin [12], and can be traced back to Sevast'yanov [16]. It differs from the ordinary Markov branching process (MBP), in that branching events are affected by the interaction/collision of pairs of particles, rather than by the particles individually. It could be used to model high-energy subatomic particle collisions, such as the proton–proton chain [7], as well as the effect of interactions between spreaders in the Daley–Kendall [8] and Maki–Thompson [14] rumour models, and between searching insect parasites (Rogers and Hassell [15]). Several extensions are possible, including branching and collision [13], and, more generally, \( k \)-particle interactions [11], but this generality is achieved at the expense of tractability.

In order that the branching property holds for the ordinary MBP it is necessary that its transition function obeys the Kolmogorov forward equations (see Asmussen and Hering [2], Athreya and Jagers [3], Athreya and Ney [4], and Harris [10]). Guided by this fact, we formally define the collision branching process as follows.

**Definition 2.** A collision branching process (CBP) is a continuous-time Markov chain, taking values in \( \mathbb{Z}_+ \), whose transition function \( P(t) = \{ p_{ij}(t), \; i, j \in \mathbb{Z}_+ \} \) satisfies the forward equation

\[ P'(t) = P(t) Q, \tag{3} \]

where \( Q \) is a CB \( q \)-matrix and a prime denotes component-wise differentiation.

Since CBPs have two absorbing states, 0 and 1, there is a need to evaluate probabilities of absorption for these states individually. Also, since the total rate out of each state \( i \) is a quadratic function of \( i \), one might expect explosive behaviour to occur more readily than for the MBP. We will examine both matters in detail. Regularity and uniqueness criteria are established in Section 2, thus extending the Sevast'yanov sufficient condition for regularity [16] (see also [12, p. 281]). In Section 3 we review the evaluation of extinction probabilities, first considered by Ezhov [9] (we note that there is an apparent typographical error in the transition rates described in [9]). Our methods are based on the forward equations. They easily generalize to the case of general weights (to be considered in a future paper), and allow us to evaluate the probability of explosion. This section concludes with a derivation of the expected hitting times for each of the absorbing states. The final section details an example that illustrates our results.

**2. Uniqueness**

General theory dictates that there always exists a CBP, namely the Feller minimal process (see, for example, Anderson [11], Chen [6], Wang and Yang [17], and Yang [18]). Under what conditions is it unique, however? In order to investigate this question, we introduce the
generating function \( B \) of the sequence \( \{b_j, j \geq 0\} \) given in (2), as follows:

\[
B(s) = \sum_{j=0}^{\infty} b_j s^j, \quad |s| \leq 1.
\]

This satisfies \( B(0) = b_0 > 0 \) and \( B(1) = 0 \). Also, \( m_1 := B'(1) = \sum_{j=1}^{\infty} j b_{j+2} - 2b_0 - b_1 \) satisfies \(-\infty < m_1 \leq +\infty\). This latter quantity measures the drift away from 0, because we see that, after normalizing by \( \sum_{j \neq 3} b_j \), \( m_1 \) is the expected jump size from any state \( i \).

The sign of \( m_1 \) determines the number of zeros of \( B \) in \([0, 1] \), as the following simple result demonstrates.

**Lemma 1.** The equation \( B'(s) = 0 \) has at most two distinct roots in \([0, 1]\). More specifically, if \( m_1 \leq 0 \) then \( B(s) > 0 \) for all \( s \in [0, 1] \) and 1 is the only root of the equation \( B(s) = 0 \) in \([0, 1] \), while if \( m_1 > 0 \) (including the case \( m_1 = +\infty \)) then \( B(s) = 0 \) has an additional root \( q \) satisfying \( 0 < q < 1 \), with \( B(s) > 0 \) for \( 0 \leq s < q \) and \( B(s) < 0 \) for \( q < s < 1 \).

**Proof.** We first prove that the equation \( B'(s) = 0 \) has either precisely one root or precisely two roots in \([0, 1]\). On the one hand, since \( B'''(s) > 0 \) for all \( s \in [0, 1] \), \( B' \) is convex on \([0, 1]\). It follows that \( B'(s) = 0 \) has at most two roots in \([0, 1]\). On the other hand, \( B'(s) = 0 \) must have at least one root in \((0, 1)\), for otherwise \( B'(s) > 0 \) for all \( s \in (0, 1) \), which would imply that \( B(s) \) is increasing on \([0, 1]\), contradicting the facts that \( B(0) = b_0 > 0 \) and \( B(1) = 0 \). Next, since \( B'(s) = 0 \) has either one root or two roots in \((0, 1)\), it is easily seen that the former is true if and only if \( B'(1) \leq 0 \), because \( B'(0) = b_1 > 0 \). Hence, if \( B'(1) \leq 0 \) there exists \( \xi \in (0, 1) \) such that \( B'(s) > 0 \) for all \( s \in (0, \xi) \) and \( B'(s) < 0 \) for all \( s \in (\xi, 1) \), implying that \( B(s) \) is strictly increasing on \([0, \xi]\) and strictly decreasing on \([\xi, 1]\). But \( B(0) = b_0 > 0 \) and \( B(1) = 0 \), and so \( B(s) > 0 \) for all \( s \) in \([0, 1]\), implying that \( 1 \) is the only root of \( B(s) = 0 \) in \([0, 1]\).

If \( B'(1) > 0 \) (including \( B'(1) = +\infty \)) then there exist \( \xi_1 \) and \( \xi_2 \) with \( 0 < \xi_1 < \xi_2 < 1 \), such that \( B'(\xi_1) = B'(\xi_2) = 0 \), that \( B'(s) > 0 \) for all \( s \in [0, \xi_1] \cup [\xi_2, 1] \), and that \( B'(s) < 0 \) for \( s \in (\xi_1, \xi_2) \). It follows that \( B(s) \) is strictly increasing on \([0, \xi_1] \cup [\xi_2, 1]\) and strictly decreasing on \([\xi_1, \xi_2]\). Thus, in addition to the root \( 1 \), \( B(s) = 0 \) must have another root in \((0, 1)\), which in fact belongs to \((\xi_1, \xi_2)\). The proof is now complete.

Later, in Lemmas 3 and 4, we will elucidate further properties of the zeros of \( B \), as they are needed. Henceforth, we will always denote by \( q \) the smallest positive zero of \( B \) on \((0, 1)\), so that \( q \) is strictly less than 1 if \( m_1 > 0 \), or equal to 1 if \( m_1 \leq 0 \).

We are now ready to settle the question of uniqueness. Recall that a conservative \( q \)-matrix \( Q \) is said to be regular if the Feller minimal \( Q \)-transition function is honest, i.e. \( \sum_{j=0}^{\infty} p_{ij}(t) = 1 \), and, when this condition holds, it is the only such transition function.

**Theorem 1.** The CB \( q \)-matrix is regular if and only if \( m_1 \leq 0 \).

**Proof.** Suppose that \( m_1 \leq 0 \) and let \( P(t) = \{p_{ij}(t)\} \) be the minimal \( Q \)-transition function. Substituting (1) into the forward equations (3) gives

\[
p'_{ij}(t) = \sum_{k=0}^{j-1} p_{ik}(t) b_{j-k+2}, \quad i, j \geq 0.
\]
Some algebra then yields, for $0 \leq s < 1$,
\begin{equation}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}, \quad i \geq 0,
\end{equation}
the right-hand side being strictly positive for $s \in (0, 1)$, by Lemma 1. Now, general theory dictates that, for all $t \geq 0$,
\begin{equation}
\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q_i,
\end{equation}
where $q_i := -q_{ii} = \binom{i}{2}b_2 < \infty$ (see, for example, Proposition 1.2.6(2) of [1]). Therefore, the series $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$ converges uniformly on $[0, \infty]$ for every $s \in [0, 1)$ and, since the derivatives $p'_{ij}(t)$ are all continuous (see, for example, Proposition 1.2.4(2) of [1]), the derivative of $\sum_{j=0}^{\infty} p_{ij}(t)s^j$ exists and equals $\sum_{j=0}^{\infty} p_{ij}(t)s^j$. Thus, we may integrate (4) to obtain
\begin{equation}
\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0, \quad i \geq 0, \quad 0 \leq s < 1.
\end{equation}
Letting $s \uparrow 1$ in (6) yields $\sum_{j=0}^{\infty} p_{ij}(t) \geq 1$, implying that equality holds for all $i \geq 0$. We deduce that the minimal $Q$-transition function is honest, and hence that $Q$ is regular.

Conversely, suppose that $m_1 > 0$. Define a (conservative) birth–death $q$-matrix $Q^* = \{q^*_{ij}, i, j \in \mathbb{Z}_+\}$ by
\begin{equation*}
q^*_{ij} = \begin{cases}
\binom{i}{2}b^* & \text{if } j = i + 1, \ i \geq 2, \\
\binom{i}{2}a^* & \text{if } j = i - 1, \ i \geq 2, \\
-\binom{i}{2}(b^* + a^*) & \text{if } j = i \geq 2, \\
0 & \text{otherwise},
\end{cases}
\end{equation*}
where $b^* > a^* > 0$. Since $\sum_{i=2}^{\infty} \binom{i}{2}^{-1} < +\infty$, Theorem 3.2.2 of [1] implies that $Q^*$ is not regular. Our aim is to choose $a^*$ and $b^*$ in such a way that a comparison of $Q^*$ with the original CB $q$-matrix $Q$ leads to the conclusion that $Q$ is not regular.

To this end, first note that $m_1 > 0$ is equivalent to $2b_0 + b_1 < \sum_{j=1}^{\infty} jb_{j+2} \leq +\infty$, and so we may choose $a^*$ and $b^*$ such that
\begin{equation}
2b_0 + b_1 < a^* < b^* < \sum_{j=1}^{\infty} jb_{j+2}.
\end{equation}
Since $Q^*$ is not regular, the equation
\begin{equation}
(\lambda I - Q^*)u = 0 \quad (\lambda > 0),
\end{equation}

where $I$ is the identity matrix, has a nontrivial (nonnegative) bounded solution, which we shall denote by $u^* = [u_i(\lambda), \ i \geq 0]$. Clearly $u^*$ depends on both $a^*$ and $b^*$. If we can choose these constants so that
\[
\lambda u^* \leq Qu^*
\] (9)
component-wise, then we may use Theorem 2.2.7(3) of [1] to deduce that $Q$ is not regular, because, after extracting the diagonal term, (9) can be written as $\sum_{j \neq i} q_{ij} u^*_j \geq (\lambda + q_i) u^*_i$, $i \in \mathbb{Z}_+$. 

We will first prove that $a^*$ and $b^*$ can be chosen so that both
\[
\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left( \frac{a^*}{b^*} \right)^{k-1} > b^*
\] (10)
and
\[
b_0 \left( \frac{b^*}{a^*} \right) + b_0 + b_1 < a^*
\] (11)
hold. Let $\{a_n\}$ be any sequence such that $a_n \downarrow \downarrow 2b_0 + b_1$. (A double arrow denotes strict monotone convergence.) Since
\[
\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left( \frac{2b_0 + b_1}{a_n} \right)^{k-1} \uparrow \uparrow \sum_{j=1}^{\infty} j b_{j+2}
\]
as $n \to \infty$, it is clear that $b^*$ may be chosen so that
\[
\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left( \frac{2b_0 + b_1}{b^*} \right)^{k-1} > b^*
\] (12)
Similarly, by considering any sequence $\{a_n\}$ with $a_n \uparrow \uparrow b^*$, it can be seen that $a^*$ may be chosen so that both (7) and (11) hold. Now, (12) holds if $2b_0 + b_1$ is replaced by (the larger) $a^*$, which is to say that (10) holds.

To prove (9), observe that the solution to (8) satisfies $u_0(\lambda) = u_1(\lambda) = 0$ and
\[
b^*(u_{i+1}(\lambda) - u_i(\lambda)) = a^*(u_i(\lambda) - u_{i-1}(\lambda)) + \lambda u_i(\lambda) \left( \frac{i}{2} \right)^{-1}, \quad i \geq 2.
\] (13)
In particular, for $i = 2$, we have $b^*(u_3(\lambda) - u_2(\lambda)) = (a^* + \lambda) u_2(\lambda) > 0$, implying that $\{u_i(\lambda)\}$ is strictly increasing in $i$ for each fixed $\lambda$. From (13) it is easily seen that, for all $k \geq 1$ and $i \geq 2$,
\[
u_{i+k}(\lambda) - u_{i+k-1}(\lambda) \geq \left( \frac{a^*}{b^*} \right)^{k-1} (u_{i+1}(\lambda) - u_i(\lambda))
\] (14)
and
\[
u_{i-1}(\lambda) - u_{i-2}(\lambda) \leq \left( \frac{b^*}{a^*} \right) (u_i(\lambda) - u_{i-1}(\lambda)).
\] (15)
Equation (9) is trivially true for $i = 0$ or $i = 1$. For $i \geq 2$, we have
\[
(Q u)_i = \left( \frac{i}{2} \right) \left( b_0 (u_{i-2}(\lambda) - u_i(\lambda)) + b_1 (u_{i-1}(\lambda) - u_i(\lambda)) + \sum_{j=i+1}^{\infty} b_{j-i+2} (u_j(\lambda) - u_{i}(\lambda)) \right)
\]
\[
= \left( \frac{i}{2} \right) (-I_d + I_b)
\] (16)
(in an obvious notation, with both \( I_d \) and \( I_b \) positive). By (10) and (14), we have that
\[
I_b \geq \sum_{j=1}^{\infty} \sum_{k=1}^{j} \left( \frac{a^*}{b^*} \right)^{k-1} (u_{j+1}(\lambda) - u_j(\lambda)) > b^*(u_{j+1}(\lambda) - u_j(\lambda)).
\] (17)

Similarly, (11) and (15) imply that
\[
I_d \leq \left( \frac{b^*}{a^*} + b_0 + b_1 \right) (u_i(\lambda) - u_{i-1}(\lambda)) < a^*(u_i(\lambda) - u_{i-1}(\lambda)).
\] (18)

In view of (17) and (18), (13) and (16) together verify (9). The converse is thus proved.

Theorem 1 establishes that if the drift \( m_1 = B'(1) \) is nonpositive, the CBP is unique. However, even when this conditional fails—and, indeed, even if \( m_1 = +\infty \) there is still only one CBP, for, as we shall see, there is a unique solution to the forward equations (3).

**Theorem 2.** There exists only one CBP.

**Proof:** As already remarked, we need only consider the case \( 0 < m_1 \leq \infty \). In order to prove that the CBP is unique, we will show that the forward equations have a unique solution. We will verify Reuter's condition (Theorem 2.2.8 of [1]) that, for some \( \lambda \), the equation \( \eta(\lambda)(\lambda I - Q) = 0, \ 0 \leq \eta(\lambda) \in l_1, \) has only the trivial solution, and then extend this to cover all \( \lambda > 0 \).

Suppose that the contrary is true, and let \( \eta = \{\eta_i, \ i \geq 0\} \) be the nontrivial solution corresponding to \( \lambda = 1 \). Then, by (1), we have that
\[
\eta_j = \sum_{i=2}^{j+2} \eta_i \binom{i}{2} b_{j-i+2}, \quad j \geq 0,
\] (19)

with
\[
\eta_j \geq 0 \quad (j \geq 0) \quad \text{and} \quad \sum_{j=0}^{\infty} \eta_j < +\infty.
\] (20)

It is clear that the nontriviality of the solution \( \eta \) implies that
\[
\sum_{j=2}^{\infty} \eta_j > 0.
\] (21)

Condition (20) guarantees that \( \sum_{j=0}^{\infty} \eta_j s^j \) is well defined, at least for all \( s \in [0, 1] \). This, in turn, implies that
\[
\sum_{j=2}^{\infty} \binom{j}{2} \eta_j s^j < +\infty, \quad 0 \leq s < 1,
\] (22)

because, by the root test, these series have the same radius of convergence. It then follows, from (19), (22), and Fubini's theorem, that \( \sum_{j=0}^{\infty} \eta_j s^j = B(s) \sum_{i=2}^{\infty} \binom{i}{2} \eta_i s^{i-2}, \ 0 \leq s < 1 \). Now, (20), (21), and (22) imply that both \( \sum_{j=0}^{\infty} \eta_j s^j \) and \( \sum_{i=2}^{\infty} \binom{i}{2} \eta_i s^{i-2} \) are strictly positive for all \( s \in (0, 1) \), and thus \( B(s) > 0 \) for all \( s \in (0, 1) \), which, since \( m_1 \in (0, \infty] \), contradicts Lemma 1. The proof is complete.
3. Extinction and explosion

Having established that the CBP is uniquely determined by its $q$-matrix, we will now examine some of its properties. Let $\{X(t), t \geq 0\}$ be the unique CBP, and let $P(t) = \{p_{ij}(t)\}$ denote its transition function. Define the extinction times $\tau_0$ and $\tau_1$ for states 0 and 1 by

$$
\tau_0 = \begin{cases} 
\inf\{t > 0 : X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0, \\
+\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0,
\end{cases}
$$

$$
\tau_1 = \begin{cases} 
\inf\{t > 0 : X(t) = 1\} & \text{if } X(t) = 1 \text{ for some } t > 0, \\
+\infty & \text{if } X(t) \neq 1 \text{ for all } t > 0,
\end{cases}
$$

and denote the corresponding extinction probabilities by

$$q^{i0} = P(\tau_0 < +\infty \mid X(0) = i) \quad \text{and} \quad q^{i1} = P(\tau_1 < +\infty \mid X(0) = i).$$

**Theorem 3.** The extinction probabilities satisfy

$$q^{i0} + q^{i1} = q^i, \quad (23)$$

where, recall, $q$ is the smallest root of $B(s) = 0$ in $[0, 1]$. More specifically,

$$q^{i0} + q^{i1} = 1 \quad \text{if } m_1 \leq 0, \quad (24)$$

$$q^{i0} + q^{i1} = q^i < 1 \quad (i \geq 1) \quad \text{if } 0 < m_1 \leq +\infty. \quad (25)$$

**Proof.** Since $q^{00} = q^{11} = 1$ and $q^{01} = q^{10} = 0$, (23) holds for $i = 0$ and $i = 1$. So, suppose that $i \geq 2$. We shall first establish (24), referring to the proof of Theorem 1. Since $m_1 \leq 0$, (6) holds. Also, $\lim_{t \to +\infty} p_{ij}(t) = 0$ for all $i, j \geq 2$, because the states $i \geq 2$ are transient. Thus, on letting $t \to +\infty$ in (6) and using the dominated convergence theorem, we find that $q^{i0} + sq^{i1} \geq s^i$ for $s \in [0, 1)$. Letting $s \uparrow 1$ leads us immediately to (24), because $q^{i0} + q^{i1} \leq 1$.

Next we will prove (25). Since $0 < m_1 \leq +\infty$, Lemma 1 implies that $q < 1$. On putting $s = q$ in (4), and noting that $B(q) = 0$, we discover that, for any $t > 0$, $\sum_{j=0}^{\infty} p_{ij}^t(0)q^j = 0$, implying that $\sum_{j=0}^{\infty} \int_0^t p_{ij}^t(u)duq^j = 0$. Hence, for any $t > 0$, we have that

$$\sum_{j=0}^{\infty} p_{ij}(t)q^j = q^i, \quad i \geq 2. \quad (26)$$

On letting $t \to +\infty$, we obtain

$$\lim_{t \to +\infty} p_{i0}(t) + \lim_{t \to +\infty} p_{i1}(t)q + \lim_{t \to +\infty} \sum_{j=2}^{\infty} p_{ij}(t)q^j = q^i, \quad i \geq 2,$$

noting that all of these limits exist. Since $q < 1$, we may apply the dominated convergence theorem in the last term on the left-hand side to obtain (25).

Theorem 3 states that if $m_1 \leq 0$ then the process is eventually absorbed at either 1 or 0 with probability 1, while if $0 < m_1 \leq +\infty$ absorption occurs with probability less than 1. Our next result establishes that if, in this latter case, absorption does not occur, then the process
must explode. In preparation, we define a family of probability generating functions $F = \{F_i(t, s), i \geq 0\}$ by $F_i(t, s) := \sum_{j=0}^{\infty} p_{ij}(t)s^j$ and note that $F$ satisfies its own set of forward equations: from (4) we get, for $s \in [0, 1)$,

$$\frac{\partial F_i(t, s)}{\partial t} = \frac{1}{2} B(s) \frac{\partial^2 F_i(t, s)}{\partial s^2}, \quad i \geq 2,$$  

(27)

with $F_0(t, s) = 1$ and $F_1(t, s) = s$.

**Lemma 2.** The transition function $P(t) = \{p_{ij}(t)\}$ satisfies

$$\lim_{t \to \infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0, \quad i \geq 2.$$  

(28)

**Proof.** Fix $i \geq 2$. First note that the limit exists because $\sum_{j=2}^{\infty} p_{ij}(t)$ is decreasing in $t$. This follows from the identity

$$p_{i0}(t) + p_{i1}(t) + \sum_{j=2}^{\infty} p_{ij}(t) = \sum_{j=0}^{\infty} p_{ij}(t),$$  

(29)

because the first two terms on the left-hand side are increasing, while the right-hand side is decreasing. Thus, we only need to prove that the limit in (28) equals 0. When $m_1 \leq 0$, $P$ is honest (Theorem 1) and $q^{i0} + q^{i1} = 1$ (Theorem 3), so letting $t \to \infty$ in (29) achieves the desired result.

Now suppose that $m_1 > 0$. Observe that (4) holds for all $s \in [0, 1)$, no matter what the value of $m_1$; when $s = q$, the right-hand side is zero. Thus, we may write

$$\frac{1}{B(s)} \sum_{j=0}^{\infty} \frac{p_{ij}'(t)}{s^j} = \sum_{k=2}^{\infty} \binom{k}{2} p_{ikk}'(t)s^{k-2},$$  

(30)

for all $s \in [0, 1)$. The apparent singularity at $s = q$ on the left-hand side is removable, because the series on the right-hand side certainly converges for all $s \in [0, 1)$. Moreover, the left-hand side is continuous and strictly positive (indeed increasing) on this interval. Therefore, on integrating (27) with respect to $s$ and using Fubini's theorem, we deduce that, for any $s \in [0, 1)$,

$$F_i(t, s) = p_{i0}(t) + p_{i1}(t) + s + \int_0^s \frac{s - y}{B(y)} F_i'(t, y) \, dy,$$  

(31)

where $F_i'(t, y) := \frac{\partial F_i(t, y)}{\partial t}$. Letting $s \uparrow 1$ shows that (31) also holds for $s = 1$, and so

$$\sum_{j=2}^{\infty} p_{ij}(t) = 2 \int_0^1 \frac{1 - y}{B(y)} F_i(t, y) \, dy.$$  

Thus, the proof will be complete if we can establish that

$$\lim_{t \to \infty} \int_0^1 \frac{1 - y}{B(y)} F_i(t, y) \, dy = 0.$$
To this end, first observe that, for $\epsilon \in (0, 1)$,

$$\lim_{t \to \infty} \int_0^{1-\epsilon} \frac{1-y}{B(y)} F_y^s(t, y) \, dy = 0,$$

since, by (30), the integrand is dominated by $1/(1-y)^2$, and because the limit as $t \to \infty$ of the left-hand side of (30) is equal to 0 for $s \in [0, 1)$. It therefore suffices to prove that

$$\lim_{t \to \infty} \int_{1-\epsilon}^{1} \frac{1-y}{B(y)} F_y^s(t, y) \, dy = 0,$$

for some suitable $\epsilon$. We will use (5), together with the fact that the root $s = 1$ of $B(s) = 0$ has multiplicity 1 when $m_1 > 0$ (because $B'(1) > 0$). In particular, (5) implies that

$$-F_y^s(t, s) = |F_y^s(t, s)| \leq \sum_{j=0}^{\infty} |p_{ij}^s(t)| s^j \leq \sum_{j=0}^{\infty} |p_{ij}^s(t)| \leq 2q_i, \quad q < s < 1,$$

remembering that $F_y^s(t, s)/B(s) > 0$ for $s \in [0, 1)$, and $B(s) < 0$ for $s \in (q, 1)$. Therefore, if we take $\epsilon < 1 - q$, we find that

$$\int_{1-\epsilon}^{1} \frac{1-y}{B(y)} F_y^s(t, y) \, dy \leq 2q_i \int_{1-\epsilon}^{1} \frac{1-y}{-B(y)} \, dy < \infty,$$

and so again dominated convergence can be used to obtain the desired result.

It is interesting to contrast the behaviour described in Theorems 1, 2, and 3, and Lemma 2, with that of the ordinary MBP. Like the CBP, the MBP is always unique and, in the subcritical and critical cases (these being analogous to our cases $m_1 < 0$ and $m_1 = 0$, respectively), absorption occurs with probability 1. However, in the supercritical case ($m_1 > 0$), the behaviour of the two processes is different: whilst both are absorbed with probability less than 1, the CBP is always dishonest (Theorem 1), whereas the MBP can only be dishonest when $m_1 = +\infty$, and this happens when and only when Harris' integral condition fails (see, for example, Theorem 3.3.3 of [1]). Lemma 2 establishes that, unlike the MBP, the CBP may never drift passively towards infinity. If absorption does not occur, the CBP will certainly explode. The latter is also true of the MBP, in that when the MBP is dishonest (i.e. $m_1 = +\infty$ and Harris' condition fails) it is either absorbed or it explodes with probability 1 (see [5]).

In order to evaluate the absorption and explosion probabilities explicitly, we will need the following result.

**Lemma 3.** The equation $B(s) = 0$ has a unique root $\xi$ in $(-1, 0)$, and this satisfies

$$q^{i0} + \xi q^{i1} = \xi^i.$$

**Proof.** Since $B(-1) < 0$ and $B(0) > 0$, we know that $B(s) = 0$ has at least one root in $(-1, 0)$. To prove uniqueness, assume that there are two distinct roots, $\xi_1$ and $\xi_2$, in $(-1, 0)$. A careful examination of the argument leading to (4) reveals that (4) also holds for all $s \in (-1, 0)$. We therefore deduce that $\sum_{j=0}^{\infty} p_{ij}(t) \xi_1^j = \xi_1^i$ and $\sum_{j=0}^{\infty} p_{ij}(t) \xi_2^j = \xi_2^i$, using the kind of argument that led to (26). Hence, following the proof of Theorem 3, we see that, for $i \geq 2$, $q^{i0} + \xi_1 q^{i1} = \xi_1^i$ and $q^{i0} + \xi_2 q^{i1} = \xi_2^i$. Letting $i = 2$, while remembering that $\xi_1 \neq \xi_2$, yields $q^{i1} = \xi_1 + \xi_2$. Since $\xi_1, \xi_2 \in (-1, 0)$, this is a contradiction. Thus, both uniqueness and (32) are proved.
Theorem 3, as well as Lemmas 2 and 3, allow us to evaluate the extinction probabilities $q^{i0}$ and $q^{i1}$, as well as $q^{i\infty}$, the probability of explosion starting in state $i$. Here and henceforth we will always use $\xi$ to denote the unique root of $B(s) = 0$ in $(-1, 0)$.

**Theorem 4.** (i) If $m_1 \leq 0$ then

\[ q^{i0} = \frac{\xi^i - \xi}{1 - \xi}, \quad q^{i1} = \frac{1 - \xi^i}{1 - \xi}, \quad \text{and} \quad q^{i\infty} = 0. \tag{33} \]

(ii) If $0 < m_1 \leq +\infty$ then

\[ q^{i0} = \frac{q^i - \xi q^i}{q - \xi}, \quad q^{i1} = \frac{q^i - \xi^i}{q - \xi}, \tag{34} \]

and $q^{i\infty} = (q(1 - \xi) - \xi(1 - q^i) - (q^i - \xi^i))/(q - \xi)$.

**Proof.** We have already noted, in the proof of Lemma 2, that when $m_1 \leq 0$ the honesty of $P$ implies that $q^{i0} + q^{i1} = 1$. On the other hand, when $m_1 > 0$ we have

\[ q^{i0} + q^{i1} + q^{i\infty} = 1, \quad i \geq 2, \]

by virtue of Lemma 2. The explicit expressions for $q^{i0}$, $q^{i1}$, and $q^{i\infty}$ are then a direct consequence of Theorem 3 and Lemma 3.

Next we will evaluate the expected hitting times. Let $\mu_{ik} = E(\tau_k 1_{\{\tau_k < \infty\}} | X(0) = i)$, $k = 0, 1$, denote the expected extinction times starting in state $i$. Similarly, let $\mu_{i\infty} = E(\tau_{\infty} 1_{\{\tau_{\infty} < \infty\}} | X(0) = i)$, where $\tau_{\infty}$ is the explosion time. The following simple result will permit us to decide whether or not these expected hitting times are finite. It determines the multiplicity of the roots $\xi$, $q$ (and 1) of the equation $B(s) = 0$ in terms of $m_1$. Recall that a root is said to be simple if it has multiplicity 1, and that $m_1 := B'(1)$.

**Lemma 4.** (i) The root $\xi$ is always simple.

(ii) If $m_1 > 0$ (including $m_1 = +\infty$) then both roots $q$ and 1 are simple.

(iii) If $m_1 < 0$ then the unique root 1 in $[0, 1]$ is simple.

(iv) If $m_1 = 0$ then the unique root 1 in $[0, 1]$ has multiplicity 2.

**Proof.** (i) If $\xi$ is not simple, then $B(\xi) = B'(\xi) = 0$. On differentiating (4) with respect to $s$ (remembering that (4) is valid for $s \in (-1, 1)$) and setting $s = \xi$, we obtain

\[ \sum_{j=1}^{\infty} p_{ij}(t) j \xi^{j-1} = 0, \quad t \geq 0, \quad i \geq 2, \]

which, in turn, gives

\[ \sum_{j=1}^{\infty} j p_{ij}(t) \xi^{j-1} = i \xi^{i-1}, \quad t \geq 0, \quad i \geq 2. \]
Since $|\xi| < 1$, and hence $\sum_{j=1}^{\infty} j|\xi|^{j-1} < \infty$, we may let $t \to \infty$ and apply the dominated convergence theorem to obtain $q^{1/2} = i\xi^{1/2}$, $i \geq 2$, which is at variance with either (33) or (35) (whichever is appropriate for the given value of $m_1$).

The proofs of (ii) and (iii) are simpler: (ii) follows because here $B'(q) < 0$ (see the proof of Lemma 1) and $B'(1) > 0$, and (iii) follows because now $B'(1) < 0$.

In case (iv), since here $B'(1) = 0$, it is sufficient to verify that $B''(1) \neq 0$. However, this is nearly obvious; since $B'(1) = 0$ is equivalent to $\sum_{j=3}^{\infty} (j-2)b_j = 2b_0 + b_1$, it follows that

$$B''(1) = \sum_{j=3}^{\infty} (j+1)(j-2)b_j - 2(b_0 + b_1)$$

$$= \sum_{j=3}^{\infty} (j-2)b_j - b_1 > \sum_{j=3}^{\infty} (j-2)b_j - b_1 = 2b_0 > 0.$$  

The proof is now complete.

**Theorem 5.** (i) If $m_1 \leq 0$, the expected extinction times are all finite and are given by

$$\mu_{i0} = \frac{2}{(1-\xi)^2} \left( -\xi \int_{0}^{1} \frac{(1-y)^2}{B(y)} f_i(y) \, dy + \int_{\xi}^{1} \frac{(y-\xi)(1-y)}{B(y)} f_i(y) \, dy \right),$$  \hspace{1cm} (36)

$$\mu_{i1} = \frac{2}{(1-\xi)^2} \left( \int_{0}^{1} \frac{(1-y)^2}{B(y)} f_i(y) \, dy - \int_{\xi}^{1} \frac{(y-\xi)(1-y)}{B(y)} f_i(y) \, dy \right),$$  \hspace{1cm} (37)

for $i \geq 2$, where

$$f_i(y) = \xi^i - \frac{\xi(1-y^i)}{1-y} + \frac{y(1-y^{i-1})}{1-y}.$$  \hspace{1cm} (38)

(ii) If $0 < m_1 \leq +\infty$ then, again, the expected extinction times are all finite. They are given by

$$\mu_{i0} = \frac{2}{(q-\xi)^2} \left( -\xi \int_{0}^{q} \frac{(q-y)^2}{B(y)} f_i(y) \, dy + q \int_{\xi}^{1} \frac{(y-\xi)(q-y)}{B(y)} f_i(y) \, dy \right),$$  \hspace{1cm} (39)

$$\mu_{i1} = \frac{2}{(q-\xi)^2} \left( \int_{0}^{q} \frac{(q-y)^2}{B(y)} f_i(y) \, dy - q \int_{\xi}^{1} \frac{(y-\xi)(q-y)}{B(y)} f_i(y) \, dy \right),$$  \hspace{1cm} (40)

for $i \geq 2$, where

$$f_i(y) = \xi^i - \frac{\xi(q^i - y^i)}{q - y} + \frac{qy(q^{i-1} - y^{i-1})}{q - y}.$$  \hspace{1cm} (41)

**Proof.** To begin with, note that all of the integrals in (36), (37), (39), and (40) are finite; since the functions defined in (38) and (41) are bounded on $[-1, 1]$ (indeed, $|f_i(y)| \leq 2i$ for $y \in [-1, 1]$), the finiteness of the integrals follows from Lemma 4.

We will prove (ii) first. Since $0 < m_1 \leq +\infty$, we have that $0 < q < 1$. Integrating (27) with respect to $s$, and using Fubini's theorem, shows that, for any $s \in [0, q]$,

$$F_i(t, s) = p_{i0}(t) + p_{i1}(t)s + 2 \int_{0}^{s} \frac{s-y}{B(y)} F_i'(t, y) \, dy,$$  \hspace{1cm} (42)

where, as before, $F_i'(t, y) = \partial F_i(t, y)/\partial t$. Similarly, integrating along the negative real axis gives, for any $s \in [\xi, 0]$,

$$F_i(t, s) = p_{i0}(t) + p_{i1}(t)s + 2 \int_{s}^{0} \frac{y-s}{B(y)} F_i'(t, y) \, dy.$$  \hspace{1cm} (43)
Next, let \( s = q \) in (42) and \( s = \xi \) in (43), and use the facts that \( \sum_{j=0}^{\infty} p_{ij}(t) q^j = q^i \) and \( \sum_{j=0}^{\infty} p_{ij}(t) \xi^j = \xi^i \), which follow from the fact that both \( q \) and \( \xi \) are roots of \( B(s) = 0 \) (again, refer to the argument leading to (26)). This gives

\[
\begin{align*}
    p_{i0}(t) + p_{i1}(t)q &= q^i - 2 \int_0^q \frac{q - y}{B(y)} f'_i(t, y) \, dy, \\
    p_{i0}(t) + p_{i1}(t)\xi &= \xi^i - 2 \int_\xi^0 \frac{y - \xi}{B(y)} f'_i(t, y) \, dy.
\end{align*}
\]

Now, in view of (23) and (32), the above equations can be rewritten as

\[
\begin{align*}
    (q^{i0} - p_{i0}(t)) + (q^{i1} - p_{i1}(t))q &= 2 \int_0^q \frac{q - y}{B(y)} f'_i(t, y) \, dy, \\
    (\xi^{i0} - p_{i0}(t)) + (\xi^{i1} - p_{i1}(t))\xi &= 2 \int_\xi^0 \frac{y - \xi}{B(y)} f'_i(t, y) \, dy.
\end{align*}
\]

Therefore, since \( q^{ik} - p_{ik}(t) = P(t < \tau_k < \infty \mid X(0) = i), k = 0, 1, \) integrating with respect to \( t \) from 0 to \( \infty \) yields

\[
\begin{align*}
    \mu_{i0} + \mu_{i1} q &= 2 \int_0^q \frac{q - y}{B(y)} (F_i(\infty, y) - F_i(0, y)) \, dy, \\
    \mu_{i0} + \mu_{i1} \xi &= 2 \int_\xi^0 \frac{y - \xi}{B(y)} (F_i(\infty, y) - F_i(0, y)) \, dy,
\end{align*}
\]

where \( F_i(\infty, y) := \lim_{t \to \infty} F_i(t, y) \). Upon using the identities \( F_i(\infty, y) = q^{i0} + q^{i1} y \) and \( F_i(0, y) = y^i \), and solving for \( \mu_{i0} \) and \( \mu_{i1} \), we eventually arrive at (39)–(41). Application of the dominated convergence theorem is permitted because both \( \int_0^q (q - y)^2 / B(y) \, dy \) and \( \int_\xi^0 (y - \xi)(q - y)/B(y) \, dy \) are finite, as noted above.

The proof of (i) is the same, except that \( q \) is replaced by 1, and application of the dominated convergence theorem is permitted because both \( \int_\xi^0 (y - \xi)(1 - y)/B(y) \, dy \) and \( \int_0^1 (1 - y)^2 / B(y) \, dy \) are finite.

Next, we will evaluate the expected time to explosion. By Theorem 4, only the case \( 0 < m_1 \leq +\infty \) need be considered. Since we are dealing with the minimal process,

\[
p_{i\infty}(t) := 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \leq t \mid X(0) = i)
\]

is the probability of explosion by time \( t \), starting in state \( i \), and \( p_{i\infty}(t) \to q^{i\infty} \) as \( t \to \infty \).

**Theorem 6.** If \( 0 < m_1 \leq +\infty \) then the expected explosion time is finite and is given by

\[
\mu_{i\infty} = \frac{2}{(q - \xi)^2} \left( (q - \xi) \int_0^1 \frac{(1 - y)(y - q)}{B(y)} f_i(y) \, dy - (1 - \xi) \int_0^q \frac{(q - y)^2}{B(y)} f_i(y) \, dy \right.
\]

\[
\left. + \left(1 - q\right) \int_\xi^0 \frac{(y - \xi)(q - y)}{B(y)} f_i(y) \, dy \right)
\]

for \( i \geq 2 \), where \( f_i(y) \) is given in (41).
Proof. Fix \( i \geq 2 \) and observe that \( \mu_{i\infty} < \infty \), since, as already noted, all of the integrals in (45) are finite. Since \( 0 < m_1 \leq +\infty \) we know that \( p_{i\infty}(t) > 0 \). Furthermore, \( \mu_{i\infty} = \int_0^\infty (q^{i\infty} - p_{i\infty}(t)) \, dt \) because \( P(t < \tau_\infty < \infty \mid X(0) = i) = q^{i\infty} - p_{i\infty}(t) \), and \( q^{i\infty} = 1 - q^{i0} - q^{i1} \), where \( q^{i0} \) and \( q^{i1} \) are given in Theorem 4(ii). This, together with (44), yields

\[
\mu_{i\infty} = \sum_{j=2}^\infty \int_0^\infty p_{ij}(t) \, dt - \mu_{i0} - \mu_{i1}, \tag{46}
\]

where \( \mu_{i0} \) and \( \mu_{i1} \) are given in (39) and (40), respectively. Note that \( \int_0^\infty p_{ij}(t) \, dt < \infty \) for all \( j \geq 2 \), because all of the states \( j \geq 2 \) are transient. Moreover, by virtue of (4), this integral can be evaluated explicitly; on integrating (4) with respect to \( t \) from 0 to \( \infty \), we get

\[
\frac{q^{i0} + q^{i1}s - s^i}{B(s)} = \sum_{k=2}^\infty \binom{k}{2} \int_0^\infty p_{ik}(t) \, dt \, s^{k-2}, \quad |s| < 1, \tag{47}
\]

and extracting the coefficient of \( s^{k-2} \) gives

\[
\int_0^\infty p_{ik}(t) \, dt = \frac{2}{k!} G_i^{(k-2)}(0), \quad i \geq 2, \quad k \geq 2, \tag{48}
\]

where

\[
G_i(s) = \frac{q^{i0} + q^{i1}s - s^i}{B(s)}. \tag{49}
\]

Now, integrating (47) twice with respect to \( s \) yields

\[
\sum_{k=2}^\infty \int_0^\infty p_{ik}(t) \, dt \, s^k = 2 \int_0^s (s - y) G_i(y) \, dy,
\]

and letting \( s \uparrow 1 \) shows that

\[
\sum_{k=2}^\infty \int_0^\infty p_{ik}(t) \, dt = 2 \int_0^1 (1 - y) G_i(y) \, dy. \tag{50}
\]

Substituting (50) into (46) then yields

\[
\mu_{i\infty} = 2 \int_0^1 (1 - y) G_i(y) \, dy - \mu_{i0} - \mu_{i1} \tag{51}
\]

and, after substituting the expressions for \( q^{i0} \) and \( q^{i1} \), given in (34) and (35), into (49), some rearrangement gives

\[
\int_0^1 (1 - y) G_i(y) \, dy = \int_0^1 \frac{(q - y)(1 - y)}{(q - \xi) B(y)} f_i(y) \, dy, \tag{52}
\]

where \( f_i(y) \) is given by (41). Finally, on substituting (52), (39), and (40) into (51), we arrive at (45).
We have proved that the CBP either explodes, or is absorbed, in finite-mean time. Our final result concerns the time spent in each state over the lifetime of the process. Let $T_k$ be the total time spent in state $k \geq 2$ and let $\mu_{ik} = E(T_k \mid X(0) = i), i \geq 2$. Then,

$$\mu_{ik} = E \left( \int_0^\infty I_{\{X(t) = k\}} \, dt \bigg| X(0) = i \right) = \int_0^\infty p_{ik}(t) \, dt.$$  

This expression was evaluated in (48). We have therefore proved the following result.

**Theorem 7.** All of $\mu_{ik}, i \geq 2, k \geq 2$, are finite and given by

$$\mu_{ik} = \frac{2}{k!} g_i^{(k-2)}(0), \tag{53}$$

where $g_i^{(k-2)}(0)$ is the $(k-2)$th derivative of $g_i$ near 0, given in (49). In particular,

$$\mu_{i2} = \frac{q_{i0}^2}{b(0)} = \frac{-q \xi (q^{i-1} - (q - 1)^{i-1})}{b_0(q - \xi)}, \quad i \geq 2.$$  

For example, $\mu_{22} = -q \xi / b_0$ and $\sum_{k=2}^\infty \mu_{ik} = \mu_{i0} + \mu_{i1} + \mu_{i\infty}$.

**Remark 1.** The argument used in proving Theorems 5–7 may, in principle, be extended to obtain results concerning the variance and the higher moments of the extinction, explosion, and total holding times. However, we shall not pursue this here.

**4. An example**

We will complete the paper by studying the upwardly skip-free case. This will serve to illustrate our results and to show that formulae such as (53) can be evaluated easily. Let $a_1, a_2,$ and $b$ be positive constants, and set $b_0 = a_2, b_1 = a_1, b_3 = b,$ and $b_j = 0$ for all $j \geq 4$. The generating function $B$ is given by

$$B(s) = a_2 + a_1 s - (a_1 + a_2 + b) s^2 + b s^3.$$  

It has three zeros: two positive ones, namely 1 and

$$\rho = \frac{1}{2b} (a_1 + a_2 + \sqrt{(a_1 + a_2)^2 + 4a_2 b}),$$

and a third,

$$\xi = \frac{1}{2b} (a_1 + a_2 - \sqrt{(a_1 + a_2)^2 + 4a_2 b}),$$

which is strictly negative.

Also, $m_1 = B'(1) = b - a_1 - 2a_2$. If $b < a_1 + 2a_2 (m_1 < 0)$ then $\rho > 1$ and $q = 1$. If $b = a_1 + 2a_2 (m_1 = 0)$ then $q = \rho = 1$ is a multiplicity-2 zero, while if $b > a_1 + 2a_2 (m_1 > 0)$ then $q = \rho < 1$. We can use Theorem 4 to evaluate the hitting probabilities. If $b \leq a_1 + 2a_2$, the extinction probabilities are given by

$$q^{i0} = \frac{\xi^i - \xi}{1 - \xi}, \quad q^{i1} = \frac{1 - \xi^i}{1 - \xi} \quad \text{(and thus } q^{i0} + q^{i1} = 1).$$
while, if \( b > a_1 + 2a_2 \), the extinction and explosion probabilities are given by

\[
q^{i0} = \frac{\rho_i^j - \xi_i^j}{\rho - \xi}, \quad q^{i1} = \frac{\rho_i^j - \xi_i^j}{\rho - \xi}, \quad \text{and} \quad q^{i\infty} = 1 - q^{i0} - q^{i1} > 0.
\]

In order to get more concrete results let us assume that the process starts in state \( i = 2 \). Using Theorems 5, 6, and 7, we obtain the following result.

**Proposition 1.** For the upwardly skip-free CBP the following are true:

(i) If \( b \leq 2a_2 + a_1 \) then \( q^{20} = -\xi \) and \( q^{21} = 1 + \xi \) with \( q^{2\infty} = 0 \), while if \( b > 2a_2 + a_1 \) then

\[
q^{20} = \frac{a_2}{b}, \quad q^{21} = \frac{a_1 + a_2}{b}, \quad \text{and} \quad q^{2\infty} = \frac{b - (a_1 + 2a_2)}{b} > 0.
\]

(ii) If \( b > 2a_2 + a_1 \) then

\[
\mu_{20} = \frac{2}{b(q - \xi)} \ln\left(\frac{(1 - \xi)^{q(1-\xi)}}{(1 - q)^{q(1-q)}}\right), \quad \mu_{21} = \frac{2}{b} \left(1 + \frac{1}{q - \xi} \ln\left(\frac{(1 - q)^{1-q}}{(1 - \xi)^{1-\xi}}\right)\right),
\]

\[
\mu_{2k} = \frac{2}{bk(k - 1)} \quad (k \geq 2), \quad \mu_{2\infty} = \frac{2}{b} \ln\left(\frac{(1 - q)^{1-q}(1+\xi)}{(1 - \xi)^{1-\xi}(1+q)}\right),
\]

and thus \( \mu_{20} + \mu_{21} + \mu_{2\infty} = \sum_{k=2}^{\infty} \mu_{2k} = 2/b \).

(iii) If \( b = 2a_2 + a_1 \) then

\[
\mu_{20} = \frac{2}{b} \ln(1 - \xi), \quad \mu_{21} = \frac{2}{b} (1 - \ln(1 - \xi)), \quad \mu_{2k} = \frac{2}{bk(k - 1)} \quad (k \geq 2),
\]

and thus \( \mu_{20} + \mu_{21} = \sum_{k=2}^{\infty} \mu_{2k} = 2/b \).

(iv) If \( b < 2a_2 + a_1 \) then

\[
\mu_{20} = \frac{2}{b(1 - \xi)} \ln\left(\frac{(1 - \xi)^{1-\xi}}{(1 - \rho)^{(1-\xi)/(1-\rho)}}\right), \quad \mu_{21} = \frac{2}{b} \left(1 + \frac{1}{1 - \xi} \ln\left(\frac{(1 - \rho)^{1/\rho - 1}}{(1 - \xi)^{1-\xi}}\right)\right),
\]

\[
\mu_{2k} = \frac{2}{bk(k - 1)\rho^{k-1}} \quad (k \geq 2),
\]

and thus

\[
\mu_{20} + \mu_{21} = \sum_{k=2}^{\infty} \mu_{2k} = \frac{2}{b} \left(1 - (\rho - 1) \ln\left(1 - \frac{1}{\rho}\right)\right)
\]  \quad (54)

Notice the simple form for the expected time spent in state \( k \) when \( b \geq 2a_2 + a_1 \), this being proportional to the reciprocal of \( \binom{k}{2} \). Notice also that the expected lifetime of the process is simply \( \sum_{k=2}^{\infty} \mu_{2k} = 2/b \). Yet the behaviour of the process in the two cases \( b = 2a_2 + a_1 \) and \( b > 2a_2 + a_1 \), which share this expected lifetime, is quite different. In the former case, the process will eventually be absorbed at either 0 or 1 \((m_1 = 0)\), while in the latter \((m_1 > 0)\) the process has a positive probability of explosion. Also, the same total lifetime \( 2/b \) comprises only \( \mu_{20} \) and \( \mu_{21} \) for the former case, but \( \mu_{20}, \mu_{21}, \) and \( \mu_{2\infty} \) for the latter. In contrast, when \( b < 2a_2 + a_1 \) \((m_1 < 0)\) the expected lifetime is strictly smaller than \( 2/b \) – see (54).
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