On the Existence of Quasi-Stationary Distributions in Denumerable R-Transient Markov Chains
Author(s): Masaaki Kijima
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ON THE EXISTENCE OF QUASI-STATIONARY DISTRIBUTIONS
IN DENUMERABLE R-TRANSIENT MARKOV CHAINS

MASAAKI KIJIMA, The University of Tsukuba, Tokyo

Abstract

Let \( \{X_n, n = 0, 1, 2, \cdots \} \) be a transient Markov chain which, when restricted to the state space \( \mathcal{N} = \{1, 2, \cdots \} \), is governed by an irreducible, aperiodic and strictly substochastic matrix \( P = (p_{ij}) \), and let \( p_{ij}(n) = P(X_n = j, X_0 = i) \). The prime concern of this paper is conditions for the existence of limits, \( q_{ij} \) say, of \( \lim_{n \to \infty} p_{ij}(n) = q_{ij} \) as \( n \to \infty \). If \( \sum_{j} q_{ij} = 1 \), the distribution \( (q_{ij}) \) is called the quasi-stationary distribution of \( \{X_n\} \) and has considerable practical importance. It will be shown that, under some conditions, if a non-negative non-trivial vector \( x = (x_i) \) satisfying \( x^T P x = x^T \) and \( \sum_{j} x_j = 1 \) exists, where \( r \) is the convergence norm of \( P \), i.e. \( r = R^{-1} \) and \( R = \sup\{x : \sum_{j} x_j < \infty\} \), and \( T \) denotes transpose, then it is unique, positive elementwise, and \( q_{ij}(n) \) necessarily converge to \( x_j \) as \( n \to \infty \). Unlike existing results in the literature, our results can be applied even to the R-null and R-transient cases. Finally, an application to a left-continuous random walk whose governing substochastic matrix is \( R \)-transient is discussed to demonstrate the usefulness of our results.

SUBSTOCHASTIC MATRIX; CONVERGENCE NORM; LEFT INVARIANT VECTOR;
CONDITIONAL TRANSITION PROBABILITY; RATIO LIMIT THEOREM; LEFT-CONTINUOUS
RANDOM WALK

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0. Introduction

We consider a discrete-time Markov chain \( \{\tilde{X}_n, n = 0, 1, 2, \cdots \} \) on the state space \( \mathcal{N} = \{0, 1, 2, \cdots \} \). It is assumed throughout the paper that \( \mathcal{N} \) consists of a single irreducible class and has period 1. Let \( T(i) \) be the first-passage time of \( \{\tilde{X}_n\} \) into state \( i \), i.e. \( T(i) = \inf\{n \mid \tilde{X}_n = i\} \) with the understanding that the infimum of an empty set is \( \infty \). Let \( \mathcal{N}_+ = \{1, 2, \cdots \} \) and suppose that the Markov chain \( \{\tilde{X}_n\} \) enters state 0 in a finite time with probability 1. The time \( T(0) \) may, however, be sufficiently long to allow \( \{\tilde{X}_n\} \) to settle down to a quasi-statistical equilibrium within this period. When this happens, one is interested in knowing the conditional limiting probabilities

\[
q_{ij} = \lim_{n \to \infty} \Pr[\tilde{X}_n = j \mid T(0) > n, \tilde{X}(0) = i], \quad i, j \in \mathcal{N}_+,
\]

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* Postal address: Graduate School of Systems Management, The University of Tsukuba, Tokyo, 3-29-1 Otsuka, Bunkyo-ku, Tokyo 112, Japan.
since \( q^j \) may well closely approximate the conditional steady state probabilities. The quantities \( (q^j) \), if they exist and \( \sum_{j=1}^\infty q^j = 1 \), are often called the quasi-stationary distribution and have a considerable practical importance.

In order to approach this problem, it is convenient to consider a transient Markov chain \( \{X_n, n = 0, 1, 2, \ldots \} \) restricted on \( \mathcal{N}^+ \). The transient Markov chain is governed by the substochastic matrix \( P = (p^j) \), which is obtained by deleting the column and row corresponding to state 0 of the governing stochastic matrix of \( \{X_n\} \). Let \( p^j(n) \) be the transition probabilities of \( \{X_n\} \), i.e.

\[
(0.2) \quad p^j(n) = \Pr[X_n = j, X_k \in \mathcal{N}^+ \text{ for } k = 0, 1, \ldots, n \mid X_0 = i], \quad i, j \in \mathcal{N}^+.
\]

Then, we are interested in the limits as \( n \to \infty \) of the probabilities

\[
(0.3) \quad q^j(n) = \frac{p^j(n)}{\sum_{j=1}^\infty p^j(n)}, \quad i, j \in \mathcal{N}^+.
\]

The existence of the conditional limiting probabilities \( q^j(n) \) has been extensively discussed (see e.g. Seneta [17] and references therein). Specifically, Vere-Jones [21] and Seneta and Vere-Jones [18] showed that if the substochastic matrix \( P \) is \( R \)-positive with \( R > 1 \) (see Section 1 below for the definition of \( R \)-positivity) then a unique vector \( x = (x_j) \) which is positive elementwise and satisfies

\[
(0.4) \quad rx^T = x^TP; \quad r = R^{-1}
\]

always exists and, if further \( x^TI < \infty \), \( q^j(n) \) in (0.3) converge to \( x_j/x^T1 \) as \( n \to \infty \), independent of the initial state \( i \). Here, \( T \) denotes the transpose and \( 1 = (1, 1, \ldots)^T \). However, the \( R \)-positivity is in general hard to verify. Also, the requirement of \( P \) being \( R \)-positive seems too restricted in applications. In fact, many problems of interest in practice have governing substochastic matrices that are not \( R \)-positive. For example, the substochastic matrix associated with the left-continuous random walk studied by Daley [3] is \( R \)-transient (see also [11], [18]). For such cases, the question of the convergence of \( q^j(n) \) has required a separate study for each case.

On the other hand, as shown in Theorem 4.1 of [18], if \( q^j(n) \) in (0.3) tend to limits \( x_j \) which form an honest distribution then the distribution \( x = (x_j) \) must satisfy (0.4) with \( r \) being replaced by some \( r' \leq r < 1 \) if necessary. Hence, it is of great interest to know under what conditions the existence of \( x \) satisfying (0.4) implies the convergence of \( q^j(n) \) to \( x_j \). Note that, if such conditions are identified, the problem of finding quasi-stationary distributions in explicit forms becomes twofold: (i) obtain the convergence norm \( r = R^{-1} \), and then (ii) solve the system of equations in (0.4). If the solution \( x = (x_j) \) has a convergent sum, then \( q^j(n) \) necessarily converge to \( x_j/x^T1 \) independent of the starting state. The purpose of this paper is to obtain conditions that are easy to verify and are applicable even to \( R \)-null and \( R \)-transient matrices, under which the quasi-stationary distribution exists.

This paper is organized as follows. In the next section, we give some definitions and notation. Some known results are reformulated so that we can apply them for subsequent
developments. In Section 2, we prove, under some conditions, the ratio limit theorem that \( \lim_{n \to \infty} p_{ik}(n)/p_{kj}(n) = x_i/x_j \) for all \( i, j, k \in \mathcal{N}_+ \) where \( x = (x_i) \) satisfies (0.4), which is of independent interest. Based on the results in Section 2, we provide our main result concerning the existence of the quasi-stationary distribution in Section 3. Some remarks regarding continuous-time Markov chains are also given. Finally, in Section 4, we explicitly obtain the quasi-stationary distributions of a left-continuous random walk to demonstrate the usefulness of our results. This section may be regarded as a complement of Daley's work [3]. Other applications of our results to phase-type queues will be reported in another paper [11].

1. Some preliminaries

Let \( P = (p_{ij}) \) be a non-negative matrix whose indices range over \( \mathcal{N}_+ \). For our purposes, we assume that \( P \) is strictly substochastic, i.e. \( \sum_{j} p_{ij} \leq 1 \) for all \( i \in \mathcal{N}_+ \) with at least one strict inequality. Throughout the paper, we denote by \( p_{ij}(n), n = 0, 1, \ldots \), the \( n \)-step transition probabilities, where \( p_{ij}(1) = p_{ij} \) and \( p_{ij}(0) = \delta_{ij} \), which are the components of \( P^n = (p_{ij}(n)) \), the \( n \)th power of \( P \). Here \( \delta_{ij} = 1 \) for \( i = j \) and \( \delta_{ij} = 0 \) for \( i \neq j \). Also, it is assumed throughout that \( P \) is irreducible and aperiodic.

Let \( p_{ij}(z) \) be the power series of \( p_{ij}(n) \), i.e.

\[
(1.1) \quad p_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}(n)z^n, \quad i, j \in \mathcal{N}_+,
\]

and let \( R_{ij} = \sup\{z \geq 0 : p_{ij}(z) < \infty\} \) be its convergence radius. It is well known that the convergence radius enjoys the so-called solidarity property. That is, \( p_{ij}(z) \) have a common convergence radius \( R = R_{ij} \) and converge or diverge together (see Vere-Jones [21]). If \( p_{ij}(R) < \infty \) then \( P \) is called \( R \)-transient while it is called \( R \)-recurrent otherwise. Note that, since \( P \) is substochastic, \( R \geq 1 \).

We next introduce the taboo probabilities \( k p_{ij}(n) \) of Chung [2]. Define \( k p_{ij}(1) = p_{ij} \) and \( k p_{ij}(n+1) = \sum_{i+k} k p_{ij}(n) p_{ij} \) for \( n = 1, 2, \ldots \). The usual convention that \( k p_{ij}(0) = \delta_{ij}(1-\delta_{ik}) \) will be used. Let

\[
(1.2) \quad k p_{ij}(z) = \sum_{n=0}^{\infty} k p_{ij}(n)z^n, \quad i, j \in \mathcal{N}_+.
\]

The power series \( k p_{ij}(z) \) converges in at least \( |z| < R \). It is easily seen that \( p_{ii}(R) = 1 \) for all \( i \in \mathcal{N}_+ \) if \( P \) is \( R \)-recurrent and \( p_{ii}(R) < 1 \) if \( P \) is \( R \)-transient. Also, \( k p_{ij}(R) < \infty \) for all states (see e.g. Seneta [17]). Further, let \( \mu_i(R) = \int p_{ji}(R- \cdot) \). For the case of \( P \) \( R \)-recurrent, \( \mu_i(R) \) converge or diverge together, thereby further classifying \( R \)-recurrent states. Namely, \( P \) is called \( R \)-positive if \( \mu_i(R) < \infty \) and \( R \)-null if \( \mu_i(R) = \infty \). Note that the \( R \)-positive, \( R \)-null, and \( R \)-transient properties are state properties of the transient Markov chain \( \{X_n\} \) governed by \( P \), and in general there is no convenient means to check which holds.

We start with a necessary and sufficient condition for the existence of a non-negative non-trivial solution to the system in (0.4). It should be noted that, if such a solution exists, it is necessarily positive elementwise (see for instance Lemma 3 of [18]). Also, \( r \) in
(0.4) is often called the convergence norm of $P$ and is considered as the denumerable counterpart of the Perron–Frobenius eigenvalue [17].

**Lemma 1.1.** The system $rx^T = x^TP$ has a non-negative non-trivial solution if and only if (i) $P$ is $R$-recurrent or (ii) when $P$ is $R$-transient, there is an infinite sequence of integers $\{n_i\}$ such that

\[ \lim_{m \rightarrow \infty} \frac{\sum_{j,k} p_{j,m}(R) p_{m,i}}{p_{n,i}(R)} = 0 \quad \text{for all } i \in \mathcal{N}_+. \]

**Proof.** Part (i) is due to Vere-Jones [21]. For (ii), the proofs of Harris [6] and Veech [20] go through with the only change that their $Q_{ij}$ are replaced by $P_{ij}(R)$ in (1.1).

**Remark 1.1.** A necessary and sufficient condition for the existence of the right-invariant solution to the system $rx = Px$ is given in Pruitt [16].

When $P$ is $R$-recurrent, Vere-Jones [21] has shown not only the existence of a solution to the system $rx^T = x^TP$ but also its uniqueness up to constant multiples. For the $R$-transient case, on the other hand, the uniqueness of the solution seems to require another concept. Let $\{Y_n, n = 0, 1, \ldots\}$ be a Markov chain on $\mathcal{N}_+$ governed by an irreducible, aperiodic and stochastic matrix $S$. For any subset $I$ of $\mathcal{N}_+$, denote by $L(I)$ and $U(I)$ the events $\lim \inf_{n \rightarrow \infty} \{Y_n \in I\}$ and $\lim \sup_{n \rightarrow \infty} \{Y_n \in I\}$, respectively. Also, denote by $\mathcal{M}$ and $\mathcal{C}$ the classes of $I$ with $\text{Pr} [U(I)] = 0$ and $I$ with $L(I) = U(I)$ almost everywhere (a.e.), respectively (we abuse the notation $\text{Pr} [\cdot]$ unless confusions occur). If $C \in \mathcal{C}$ and $C \notin \mathcal{M}$, $C$ is called almost closed (see Blackwell [1]). The almost closed set $C$ is called atomic if it does not contain two disjoint almost closed subsets. If $\mathcal{N}_+$ consists of a single almost closed set then the Markov chain $\{Y_n\}$ (or the matrix $S$ according to convenience) is called simple, and a simple process is atomic according to the type of its state space. The next lemma summarizes Blackwell’s results [1].

**Lemma 1.2.** Let $\{Y_n\}$ be the Markov chain stated above with the governing stochastic matrix $S$.

1. For any invariant event $V$, there is a $C \in \mathcal{C}$ such that $U(C) = V$ a.e. Here, a Borel measurable function $f$ is invariant if for every $\omega = (y_0, y_1, \ldots)$, $f(\omega) = f(\mathcal{U} \omega)$, where $\mathcal{U}(y_0, y_1, \ldots) = (y_1, y_2, \ldots)$, and an event $V$ is invariant if its characteristic function is invariant.

2. There is a finite or countable collection $\{C_1, C_2, \ldots\}$ of essentially unique disjoint almost closed sets with the following properties:

   (a) every $C_i$ except at most one is atomic,

   (b) the non-atomic $C_i$, if present, contains no atomic subsets,

   (c) $\Sigma_i \text{Pr}[L(C_i)] = 1$.

3. The process $\{Y_n\}$ is simple and atomic if and only if the only bounded solution of $Sy = y$ is $y = c1$ for $c$ constant.
Before ending this section, we give a simple sufficient condition for the process \{Y_n\} to be simple and atomic. It will be useful in applications, because it is usually not trivial to verify this property.

**Lemma 1.3.** Let \(S = (s_n)\) be irreducible, aperiodic and stochastic. If there is \(\delta > 0\) such that \(s_{i+1} \geq \delta\) for all \(i \in \mathcal{N}^+_+\), then \(S\) is simple and atomic.

**Proof.** Let \(\{Y_n\}\) be the Markov chain governed by \(S\) and suppose first, to the contrary, that \(\{Y_n\}\) is not simple. Then, from Lemma 1.2(2a), there is an atomic almost closed set \(C\). Let \(C_1 = \mathcal{N}_+ \setminus C\) and \(\partial C = \{i \in C_1 : i - 1 \in C\}\). Now, suppose \(\Pr[Y_n \in \partial C \text{ i.o.}] > 0\) (i.o. stands for infinitely often). Then, since \(\Sigma_{j \in C} p_{ij} \geq \delta\) for all \(i \in \partial C\), one has

\[
\text{(1.4)} \quad \Pr[Y_n \in C \text{ i.o.} \mid Y_n \in \partial C \text{ i.o.}] = 1.
\]

Hence, since \(C\) is almost closed so that

\[
\Pr[Y_n \in C \text{ for all sufficiently large } n \mid Y_n \in C \text{ i.o.}] = 1,
\]

one must have \(C \supset \partial C\), which contradicts \(C \cap \partial C = \emptyset\). Therefore, \(\Pr[Y_n \in \partial C \text{ i.o.}] = 0\).

Next, consider the set \(C \cup \partial C\); we show that this set is almost closed and atomic. Evidently,

\[
\{Y_n \in C \cup \partial C \text{ i.o.} = \{Y_n \in C \text{ i.o.} \cup \{Y_n \in \partial C \text{ i.o.}\}.
\]

It follows from the fact that \(\Pr[Y_n \in \partial C \text{ i.o.}] = 0\) that \(\cup (C \cup \partial C) = \cup (C)\) and \(C \cup \partial C\) can be regarded as an almost closed set. Since any non-atomic almost closed set cannot contain atomic subsets (cf. Lemma 1.2(2b)), \(C \cup \partial C\) must be atomic. Repeating the arguments until \(C_1\) becomes empty, one concludes that \(\{Y_n\}\) is simple and atomic.

Second, suppose that \(\{Y_n\}\) is simple but not atomic. Then, \(\mathcal{N}_+\) contains more than two disjoint almost closed subsets. Let \(C\) be one of the almost closed subsets and let \(C_1 = \mathcal{N}_+ \setminus C\). Then, the same arguments as above go through and we conclude that \(\mathcal{N}_+\) contains a single set, i.e. \(\mathcal{N}_+\) is atomic. This completes the proof of the lemma.

**Remark 1.2.** Variations and extensions of Lemma 1.3 can be proved similarly. For instance, if \(s_i \geq \delta\) for all \(i \in \mathcal{N}_+\), then \(S\) is simple and atomic. Hence, the process considered in Theorem 3 of Blackwell [1] is simple and atomic. This fact was proved by a different means there.

2. Some ratio limit theorems

In this section, we discuss the convergence of the ratios \(p_{\mu}(n)/p_{\mu}(n)\) with respect to \(n\). In the next section, we will see that a direct application of the ratio limit result yields the desired convergence of \(q_{ij}(n)\) in (0.3) to the quasi-stationary distribution.

When \(P\) is \(R\)-positive with \(R > 1\), it is known that the ratio limits are equal to \(x_k/x_i\) where \(x = (x_i)\) is the unique left-invariant vector of \(P\) as in (0.4) (see e.g. Theorem 6.5 on p. 207 of Seneta [17]). When \(P\) is \(R\)-null or \(R\)-transient, however, the problem is more involved. When \(P\) is stochastic and null-recurrent, Freedman [5] showed that
\[
\lim_{n \to \infty} \frac{p_{jk}(n)}{p_{ji}(n)} = \frac{x_k}{x_i}
\]

where \( x^T = x^T P \) with \( x = (x_i) \) under the following assumption.

(AS): there exists a positive integer \( N \) such that \( p_{ii}(N) \geq \delta \) for all \( i \in \mathcal{N}_+ \) and for some \( \delta > 0 \).

As shown in [5], (AS) is essential for the case that \( P \) is null-recurrent. The main results of this section are that the above ratio limit theorem holds for \( P \) being \( R \)-null under (AS) (see Remark 2.2) and for \( P \) being \( R \)-transient under (AS) with some further assumptions (see Theorem 2.1).

In what follows, we assume \( N = 1 \) in (AS) unless stated otherwise. For general \( N \), see Remark 2.1 below.

**Lemma 2.1.** Suppose (AS) holds. Then, \( \lim_{n \to \infty} [p_{ii}(n)]^{1/n} \) exist and are greater than or equal to \( \delta > 0 \) for all \( i \in \mathcal{N}_+ \).

**Proof.** The convergence of \( [p_{ii}(n)]^{1/n} \) with respect to \( n \) is proved in Lemma 2.50 of Freedman [5]. By the Chapman–Kolmogorov equations, \( p_{ii}(n+1) \geq p_{ii}(n) p_{ii} \) so that, by induction, \( [p_{ii}(n)]^{1/n} \geq p_{ii} \geq \delta \).

**Lemma 2.2.** Let \( r \) be the convergence norm of \( P \). Then, \( \lim_{n \to \infty} [p_{ij}(n)]^{1/n} = r \) for all \( i, j \in \mathcal{N}_+ \).

**Proof.** It is proved in Lemma 2.51 of [5] that there is an \( L \) such that \( 0 \leq L \leq 1 \) and \( \lim_{n \to \infty} [p_{ij}(n)]^{1/n} = L \) for all \( i, j \in \mathcal{N}_+ \), as far as \( P \) is irreducible and aperiodic. Thus, it suffices to show that \( r = L \). Write \( p_{ij}(n) = t_{ij}(n) L^n \). It follows that \( \lim_{n \to \infty} [t_{ij}(n)]^{1/n} = 1 \).

Consider then the power series \( P_{ij}(z) \) in (1.1). By the definition,

\[
(2.1) \quad P_{ij}(z) = p_{ij}(0) + \sum_{n=1}^{\infty} [t_{ij}(n)]^{1/n} zL^n.
\]

Hence, if \( \lim_{n \to \infty} [t_{ij}(n)]^{1/n} zL > 1 \) then \( P_{ij}(z) = \infty \), while \( P_{ij}(z) < \infty \) if the limit is less than 1. This means that, by the definition of \( R \), \( \lim_{n \to \infty} [t_{ij}(n)]^{1/n} RL = 1 \). Therefore, \( L = R^{-1} = r \), since \( \lim_{n \to \infty} [t_{ij}(n)]^{1/n} = 1 \).

From Lemma 2.2, one sees that, for each pair of \( i \) and \( j \) in \( \mathcal{N}_+ \), there exists a non-negative sequence \( \{t_{ij}(n)\} \) such that \( p_{ij}(n) = t_{ij}(n) r^n \) and \( \lim_{n \to \infty} [t_{ij}(n)]^{1/n} = 1 \).

Suppose, at this point, that \( P \) has a positive left-invariant vector \( x = (x_i) \) associated with \( r \), i.e. \( rx^T = x^T P \). Let \( x_D \) be the diagonal matrix whose diagonals are \( x_i > 0 \). The matrix

\[
(2.2) \quad S = (s_{ij}) = Rx_D^{-1} P^T x_D; \quad s_{ij} = R \frac{x_j}{x_i} p_{ji} = \frac{x_j}{x_i} t_{ji}
\]

is then well defined. It should be noted that \( S \) is stochastic, irreducible and aperiodic.

Also, \( S \) satisfies (AS) as far as does \( P \), since the diagonals of \( S \) agree with those of \( P \). Let \( \{Y_n\} \) be the Markov chain defined on \( \mathcal{N}_+ \) governed by the stochastic matrix \( S \). The \( n \)-step transition probabilities are denoted by \( s_{ij}(n) \). Let \( f_{ij}(n) = \Pr[T_Y(j) = n \mid Y(0) = i] \),

\[
\lim_{n \to \infty} p_{jk}(n)/p_{ji}(n) = x_k/x_i
\]
On the existence of quasi-stationary distributions

\( n = 1, 2, \ldots \), where \( T_y(j) \) denotes the first-passage time of \( \{Y_n\} \) into state \( j \). Define 
\[
\gamma^*_y = \sum_{n=1}^{\infty} f^*_y(n).
\]
Since \( S \) is irreducible, \( f^*_y > 0 \) for all \( i, j \in \mathcal{N}^+ \). Also, \( P \) is \( R \)-transient if and only if \( S \) is transient and, for this case, \( f^*_y < 1 \) for all states. Otherwise, \( S \) is recurrent and \( f^*_y = 1 \) (see e.g. Seneta [17]). As in (2.2), one easily sees that

\[
s_y(n) = R^n \frac{x_j}{x_i} p_y(n) = \frac{x_j}{x_i} t_y(n).
\]

**Lemma 2.3.**

\[
\lim_{n \to \infty} [s_y(n)]^{1/n} = 1 \quad \text{for all } i, j \in \mathcal{N}^+.
\]

**Proof.** This is immediate from the definition of \( t_y(n) \) and (2.3).

To prove the next lemma, (AS) is essential.

**Lemma 2.4.** Under (AS),

\[
\lim_{n \to \infty} \frac{s_y(n + 1)}{s_y(n)} = 1 \quad \text{for all } i, j \in \mathcal{N}^+.
\]

**Proof.** The proof proceeds exactly as in Lemma 2.55 of Freedman [5], where the case that \( S \) is recurrent is proved. It is, however, evident that the key fact there is not the recurrence but \( \lim_{n \to \infty} [s_y(n)]^{1/n} = 1 \). This condition is satisfied in the present case from Lemma 2.3.

**Corollary 2.1.** Under (AS),

\[
\lim_{n \to \infty} \frac{p_y(n + 1)}{p_y(n)} = r \quad \text{for all } i, j \in \mathcal{N}^+.
\]

**Proof.** This follows from (2.3) and Lemma 2.4.

The following ratio limit result is of independent interest.

**Lemma 2.5.** Suppose that (AS) holds and \( S \) is simple and atomic. Further suppose that the system \( Sy \leq y, y > 0 \), does not admit any unbounded solution. Then,

\[
\lim_{n \to \infty} \frac{s_k(n)}{s_y(n)} = 1 \quad \text{for all } i, j, k \in \mathcal{N}^+.
\]

**Proof.** Fix \( j \in \mathcal{N}^+ \) arbitrary. For any subsequence of \( \{n\} \), use the diagonal argument to find a sub-subsequence \( \{n'\} \) such that \( s_y(n')/s_y(n') \) converge as \( n \to \infty \), say to \( \sigma_i \), for all \( i \in \mathcal{N}^+ \) (cf. p. 68 of [5]). It should be noted that \( \sigma_i \geq 0 \) and \( \sigma_i = 1 \). Now, from Lemma 2.4,

\[
\sigma_i = \lim_{n \to \infty} \frac{s_y(n' + 1)}{s_y(n')} = \lim_{n \to \infty} \sum_{k=1}^{\infty} s_y(n') \frac{s_y(n')}{s_y(n')}.
\]

It follows from Fatou's lemma that
(2.6) \[ \sigma_i \geq \sum_{k=1}^{\infty} s_{ik} \sigma_k, \quad \sigma \geq S \sigma; \quad \sigma = (\sigma) > 0. \]

But, by the assumption, the system (2.6) does not admit any unbounded solution. Hence, \( \sigma \) are bounded and so, by the dominated convergence theorem, (2.6) holds in fact by an equality. Therefore, from Lemma 1.2(3), the system (2.6) has the unique solution \( \mathbf{1} = (1, 1, \cdots)^T \) since \( S \) is simple and atomic and \( \sigma_j = 1 \). This proves the lemma.

**Remark 2.1.** A sufficient condition for Lemmas 2.4 and 2.5 and Corollary 2.1 to hold for general \( N \) is that \( S^N \) is simple and atomic together with the conditions there. If so, similar arguments to Freedman [5] go through to assure the desired results. It should be noted that \( S^N \) may not be simple and atomic even if \( S \) is so. For example, consider the random walk example (B) with \( p > 1/2 \) in p. 73 of Karlin and McGregor [8]. For this walk, the associated stochastic matrix, say \( P \), satisfies the conditions in Lemma 1.3 so that \( P \) is simple and atomic. However, it is not difficult to show that \( P^2 \) has two almost closed sets. Therefore, even though the condition (AS) holds with \( N = 2 \) for this case, the results mentioned above may not hold. In fact, the sequence \( p_{00}(n + 1)/p_{00}(n) \) has two accumulation points, 1 and \( 4p/(1 - p) \).

We can now state the main result of this section.

**Theorem 2.1.** Let \( r \) be the convergence norm of a substochastic matrix \( P \) and suppose that there is a positive vector \( x = (x_i) \) satisfying \( rx^T = x^T P \). Further suppose that the conditions in Lemma 2.5 for the transformed matrix \( S \) in (2.2) hold. Then,

\[
(2.7) \quad \lim_{n \to \infty} \frac{p_{jk}(n)}{p_{ik}(n)} = \frac{x_k}{x_i} \quad \text{for all } i, j, k \in V_+, \]

and the left-invariant vector \( x \) is unique up to constant multiples.

**Proof.** This is immediate from (2.3) and Lemma 2.5.

The condition that the system \( Sy \leq y, y > 0 \), does not admit any unbounded solution is perhaps not clear at first sight. One may regard it as being too strong to be of theoretical interest. However, for such stochastic matrices as those in Lemma 1.3, this condition holds.

**Lemma 2.6.** Let \( S = (s_{ij}) \) be the stochastic matrix as defined in Lemma 1.3. If \( S \) is transient, the system \( Sy \leq y, y > 0 \), does not admit any unbounded solution.

**Proof.** First, we suppose that \( \lim \inf_{i \to \infty} y_i < \infty \) while \( \lim \sup_{i \to \infty} y_i = \infty \). Then, for sufficiently large \( M \), there is a subsequence \( V = \{n' \} \) such that \( y_{n'} \) diverges as \( n \to \infty \) and \( y_{n' + 1} \leq M \). Since \( \Sigma_{j=1}^{\infty} s_{ij} y_j \leq y_i \), it follows that \( s_{i+1,j} y_i \leq y_{i+1} \leq M \) for all \( i \in V \). But, by the assumption, \( s_{i+1,j} \geq 0 \) and \( y_i \) diverges as \( i \to \infty \) in \( V \), yielding a contradiction. Hence, \( \lim \inf_{i \to \infty} y_i \) should also diverge. It then follows that, for any \( \epsilon > 0 \), there is some \( N \) such that \( y_i > 2/\epsilon \) for all \( i > N \). Now,
\[ \sum_{j=1}^{N} s_{ij}(m) y_j + \sum_{j=N+1}^{\infty} s_{ij}(m) y_j \leq y_i, \]

and hence, since \( S \) is stochastic,

\[ (2.8) \]

\[ \sum_{j=1}^{N} s_{ij}(m) y_j + \frac{2}{e} \left( 1 - \sum_{j=1}^{N} s_{ij}(m) \right) \leq y_i. \]

Also, since \( S \) is transient, one can choose \( m \) sufficiently large such that \( \sum_{j=1}^{N} s_{ij}(m) < 1/2 \). Hence, from (2.8), one has \( 1/e \leq y_i \). Since \( e > 0 \) is arbitrary, this is a contradiction for \( i \leq N \), and the lemma follows.

**Remark 2.2.** When \( P \) is \( R \)-null, the result in Lemma 2.5 (and thus the results in Theorem 2.1) holds under (AS) for general \( N \). The proof proceeds as follows. First note that

\[ s_{ij}(n) \geq \sum_{m=1}^{n} f_{ij}(m) s_{ij}(n-m), \quad n = 1, 2, \ldots. \]

Hence, for each fixed \( N \) sufficiently large, Lemma 2.4 shows that

\[ \liminf_{n \to \infty} \frac{s_{ij}(n)}{s_{ij}(n)} \geq \sum_{m=1}^{N} f_{ij}(m) \lim_{n \to \infty} \frac{s_{ij}(n-m)}{s_{ij}(n)} = \sum_{m=1}^{N} f_{ij}(m). \]

It follows that \( \liminf_{n \to \infty} \{ s_{ij}(n)/s_{ij}(n) \} \geq f_{ij}^{\#} \), and hence

\[ \limsup_{n \to \infty} \frac{s_{ij}(n)}{s_{ij}(n)} \leq (f_{ij}^{\#})^{-1}. \]

But, since \( f_{ij}^{\#} = 1, i, j \in \mathcal{N}^+, \) for the recurrent case, the dominated convergence theorem assures (2.6) to hold by an equality. Also, the recurrence implies that \( S \) is simple and atomic. Hence, from Lemma 1.2(3), the result follows.

### 3. A sufficient condition for the existence of quasi-stationary distributions

Let \( T(0) \) be the first-passage time into state 0 as defined in the introductory section, and define

\[ p_{i0}(n) = \Pr[T(0) \leq n \mid \hat{X}(0) = i], \quad n = 1, 2, \ldots. \]

Note that \( p_{i0}(n) \leq 1 \) and is monotonically non-decreasing in \( n \). Hence, the limits \( p_{i0}(\infty) = \lim_{n \to \infty} p_{i0}(n) \) exist and \( p_{i0}(\infty) \leq 1 \). If \( p_{i0}(\infty) = 1 \) for all \( i \in \mathcal{N}^+ \), we say that absorption at 0 is certain. Otherwise, the process \( \{\hat{X}_n\} \) will never enter state 0 with positive probability. Also,

\[ p_{i0}(1) = 1 - \sum_{k=1}^{\infty} p_{ik} \overset{\text{def}}{=} p_{i0}, \quad i \in \mathcal{N}^+. \]

By assumption, \( p_{i0} \geq 0 \) and at least some of them are strictly positive.

We first recognize some conditions that assure the certain absorption at state 0.
Lemma 3.1. Suppose \( r < 1 \) and suppose there is a non-negative non-trivial vector \( x = (x_i) \) satisfying \( rx^T = x^T P \). If \( \Sigma_{i=1}^\infty x_i < \infty \) then absorption at state 0 is certain.

**Proof.** Let \( P_{ik}(z) \) be as defined in (1.1). It is easy to see from Fubini's theorem that

\[
(3.3) \quad P_{ik}(z) = p_{ik}(0) + z \sum_{l=1}^\infty P_{il}(z) p_{lk}, \quad |z| < R.
\]

Hence, if \( R = r^{-1} > 1 \), \( P_{ik}(1) \), \( i, k \in \mathcal{N}_+ \), exist and satisfy

\[
(3.4) \quad P_{ik}(1) = p_{ik}(0) + \sum_{l=1}^\infty P_{il}(1) p_{lk}.
\]

Also, by induction,

\[
(3.5) \quad p_{i0}(n) = p_{i0}(n-1) + \sum_{k=1}^\infty p_{ik}(n-1) p_{k0} = \sum_{j=0}^{n-1} \sum_{k=1}^\infty p_{ik}(j) p_{k0}
\]

so that, as \( n \to \infty \),

\[
(3.6) \quad p_{i0}(\infty) = \sum_{j=0}^\infty \sum_{k=1}^\infty p_{ik}(j) p_{k0} = \sum_{k=1}^\infty P_{ik}(1) p_{k0}.
\]

Now, since \( r^j x_k = \Sigma_{i=1}^\infty x_i p_{ik}(j) \), one has

\[
\sum_{j=0}^{n-1} r^j x_k = \sum_{j=0}^{n-1} \sum_{i=1}^\infty x_i p_{ik}(j), \quad n = 1, 2, \ldots.
\]

By letting \( n \) go to infinity, it follows that

\[
\frac{1}{1 - r^k} x_k = \sum_{j=0}^\infty \sum_{i=1}^\infty x_i p_{ik}(j) = \sum_{i=1}^\infty x_i P_{ik}(1).
\]

Hence, if \( \Sigma_{i=1}^\infty x_i < \infty \), one has \( \Sigma_{i=1}^\infty x_i P_{ik}(1) < \infty \) so that, using (3.2) and (3.6),

\[
(3.7) \quad \sum_{i=1}^\infty x_i p_{i0}(\infty) = \sum_{k=1}^\infty \sum_{i=1}^\infty x_i P_{ik}(1) p_{k0}
\]

\[
= \sum_{i=1}^\infty x_i \sum_{k=1}^\infty P_{ik}(1) - \sum_{i=1}^\infty x_i \sum_{j=1}^\infty \sum_{k=1}^\infty P_{ik}(1) p_{kj} < \infty.
\]

But, from (3.4), the last term in (3.7) is equal to

\[
\sum_{i=1}^\infty x_i \sum_{j=1}^\infty (P_{ij}(1) - \delta_{ij}) = \sum_{i=1}^\infty x_i \sum_{j=1}^\infty P_{ij}(1) - \sum_{i=1}^\infty x_i.
\]

Therefore, one has

\[
(3.8) \quad \sum_{i=1}^\infty x_i (p_{i0}(\infty) - 1) = 0.
\]
But, since \( p_{i0}(\infty) \leq 1 \) and one can assume \( x_i > 0 \) (see Lemma 3 of [18]), it must hold from (3.8) that \( p_{i0}(\infty) = 1 \) for all \( i \), completing the proof.

We are now in a position to state our main theorem.

**Theorem 3.1.** Let \( q_0(n) \) be as in (0.3). Under the conditions of Theorem 2.1 and Lemma 3.1, if there is an \( M \geq 1 \) such that \( p_{i0} = 0 \) for all \( i > M \), then

\[
\lim_{n \to \infty} q_0(n) = \frac{x_j}{\sum x_k} > 0, \quad \text{for all } i, j \in \mathcal{N}_+,
\]

independent of the starting state \( i \).

**Proof.** Since absorption at state 0 is certain, one has

\[
\sum_{k=1}^{\infty} p_{ik}(n) = 1 - p_{i0}(n) = p_{i0}(\infty) - p_{i0}(n).
\]

Hence, from (3.5), (3.6) and the assumptions,

\[
\sum_{k=1}^{\infty} p_{ik}(n) = \sum_{k=1}^{M} \sum_{j=0}^{\infty} p_{ik}(n+j) p_{kj}.
\]

We claim that

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{p_{il}(n+j)}{p_{ik}(n)} = \frac{1}{1 - r x_k}, \quad i, k, l \in \mathcal{N}_+.
\]

If (3.11) holds, then the theorem follows at once, since \( \Sigma_{i=1}^{\infty} x_i p_{i0} = (1 - r) \Sigma_{i=1}^{\infty} x_i \).

To prove (3.11), let \( \epsilon > 0 \) such that \( r + \epsilon < 1 \). For each \( l \), there is an \( N \) such that \( p_{il}(n+1)/p_{il}(n) < r + \epsilon \) for all \( n \geq N \), from Corollary 2.1. Hence, \( p_{il}(n+j) < p_{il}(n)(r + \epsilon)^j \) for \( n \geq N \), so that

\[
\sum_{j=0}^{\infty} \frac{p_{il}(n+j)}{p_{ik}(n)} < \frac{p_{il}(n)}{p_{ik}(n)} \sum_{j=0}^{\infty} (r + \epsilon)^j = \frac{1}{1 - r - \epsilon} \frac{p_{il}(n)}{p_{ik}(n)}.
\]

It follows from Theorem 2.1 that

\[
\lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{p_{il}(n+j)}{p_{ik}(n)} \leq \frac{1}{1 - r x_k},
\]

since \( \epsilon > 0 \) is arbitrary. The reversed inequality follows similarly. Hence, (3.11) holds and the proof is complete.

**Remark 3.1.** Theorem 3.1 with \( M = \infty \) has been proved for \( P \) being \( R \)-positive without assumption (AS) (see e.g. [17], [18], [21]). For the other cases, however, (AS) seems essential as is seen in Freedman [5].

**Remark 3.2.** In Theorem 4.1 of Seneta and Vere-Jones [18], they showed that, for \( P \) having \( 0 < r < 1 \), if \( q_0(n) \) in (0.3) converge to limits \( q_i \) that form an honest
distribution, then \( q = (q_i) \) satisfies (0.4). In the present case, from Lemmas 2.1 and 2.2, one has \( r > 0 \) under (AS). Also, it is readily shown that if (0.4) holds with \( x^T 1 = 1 \) then \( r < 1 \). Thus, the existence of a left-invariant vector satisfying (0.4) is necessary for the existence of the quasi-stationary distribution. Theorem 3.1 states its converse under the assumptions stated in the theorem. The existence of a left-invariant vector can be verified by Lemma 1.1.

To find the quasi-stationary distribution \( q = (q_i) \) corresponding to a substochastic matrix \( P \) explicitly, the problem now consists of three parts. According to Theorem 3.1, we need to (i) find the convergence norm \( r \), (ii) solve the system of equations \( rx^T = x^TP \) and (iii) check if \( S \) in (2.2) satisfies the required conditions. Then, the normalized \( q = x/x^T 1 \) is the desired distribution, provided that \( x^T 1 < \infty \). Of course, in general, the tasks (i)–(iii) above still involve many difficulties. However, this method often works for problems of interest in applications. In the next section, we demonstrate how to apply our method to obtain the quasi-stationary distribution of a left-continuous random walk considered in Daley [3]. Compared to his, our method is systematic and considerably simpler.

**Remark 3.3.** Let \( \{\tilde{X}(t), t \geq 0\} \) be a continuous-time irreducible Markov chain on \( \mathcal{N} \) and let \( \{X(t)\} \) be the corresponding transient Markov chain restricted on \( \mathcal{N}_+ \). If \( \{\tilde{X}(t)\} \) has the standard transition probabilities, its infinitesimal generator \( \tilde{Q} \) completely determines the statistical behavior of \( \{\tilde{X}(t)\} \) (see Chung [2]). Let \( Q = (q_{ij}) \) be the corresponding submatrix restricted on \( \mathcal{N}_+ \). The matrix \( Q \) is called uniformizable if there is some \( v_0 > 0 \) such that \( |q_{ii}| \leq v_0 < \infty \) for all \( i \in \mathcal{N}_+ \). If so, it is not difficult to show that the matrix defined by \( P_v = I + (1/v)Q \), where \( I \) is the identity matrix, is substochastic and satisfies (AS) for \( v > v_0 \). Since the stochastic results in the discrete-time Markov chain governed by \( P_v \) are transferred to those for the continuous-time counterpart, the quasi-stationary distribution of \( \{X(t)\} \) is determined as the quasi-stationary distribution corresponding to \( P_v \). Note that continuous-time finite Markov chains are always uniformizable [9]. Hence, quasi-stationary distributions of continuous-time finite Markov chains (in fact, of uniformizable semi-Markov processes, see Kijima [9]) can be derived through the above technique (cf. Darroch and Seneta [4]). To the author's best knowledge, very little is known for general results in quasi-stationary distributions of non-uniformizable continuous-time Markov chains (cf. [15], for example). Only the quasi-stationary distribution for birth-death processes seems to have been extensively studied (see for example [12], [14], [19] and references therein).

4. Application to left-continuous random walks

In this section, we apply Theorem 3.1 to a left-continuous random walk to obtain the quasi-stationary distribution. Other applications in the queueing context will be discussed in a separate paper [11] (see also Iglehart [7], Kyprianou [13] and references therein for the quasi-stationary distributions in queues).

Consider a left-continuous random walk \( \{\tilde{X}_n; n = 0, 1, \cdots\} \) on the integers \( \mathcal{N} \), where state 0 is absorbing, governed by a sequence \( \{a_k\}_{k=0}^\infty \) where
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\[ a_k = \Pr[\tilde{X}_{n+1} = i - 1 + k \mid \tilde{X}_n = i], \quad i \in \mathcal{N}_+ \]

such that

\[(4.1) \quad a_0 > 0, \quad a_0 + a_1 < 1 \quad \text{and} \quad \sum_{k=0}^{\infty} a_k = 1.\]

The first two conditions in (4.1) are required to eliminate trivial cases. Let \( A(z) = \sum_{k=0}^{\infty} a_k z^k \) and assume that \( A(z) \) is analytic at \( z = 1 \). Note that, since

\[(4.2) \quad m = -a_0 + \sum_{k=1}^{\infty} (k-1)a_k = A'(1) - 1 \]

represents the mean step-length of the walk, \( A'(1) < 1 \) if and only if the random walk has a negative drift. In what follows, we assume that \( a_1 > 0 \) for simplicity. It is not hard to prove the following result for the case \( a_1 = 0 \) under aperiodicity.

Let \( \{X_n\} \) be the corresponding transient Markov chain restricted on \( \mathcal{N}_+ \). The transient chain is governed by the strictly substochastic matrix

\[
A = \begin{bmatrix}
a_1 & a_2 & a_3 & \cdots \\
a_0 & a_1 & a_2 & \cdots \\
0 & a_0 & a_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

It should be noted that, from Lemma 1.1, the system \( rx^T = x^TA \) has a positive solution.

Let \( S \) be as in (2.2). To prove that \( S \) satisfies the required conditions, we first claim that there exists some \( \delta > 0 \) such that

\[(4.3) \quad \inf_{i \geq 1} \frac{x_{i-1}}{x_i} \geq \delta, \]

where \( x = (x_i) \) is a solution of \( rx^T = x^TA \). To see this, suppose (4.3) is not true. Then, for any \( \epsilon > 0 \), there is a subsequence \( \{n'\} \) such that \( x_{n'-1}/x_{n'} < \epsilon \). But, by definition,

\[
(4.4) \quad s_{n'-1,n'} = Ra_0 \frac{x_{n'}}{x_{n'-1}} > \frac{Ra_0}{\epsilon}.
\]

Since \( \epsilon > 0 \) is arbitrary, (4.4) yields \( s_{n'-1,n'} > 1 \), which is a contradiction. Thus, (4.3) holds, as claimed. Next, we claim that there exists some \( \delta' > 0 \) such that

\[(4.5) \quad \inf_{i \geq 1} \frac{x_{i+1}}{x_i} \geq \delta'. \]

For, if not, there is a subsequence \( \{n'\} \) such that \( x_{n'+1}/x_{n'} < \epsilon \) for any \( \epsilon > 0 \). But, since there is some \( n \geq 2 \) such that \( a_n > 0 \), one has \( x_{n'+n-1} \equiv a_nx_{n'} \) so that, from (4.3),

\[
a_n \leq \frac{x_{n'+1}}{x_{n'}} \frac{x_{n'+n-1}}{x_{n'+1}} < \epsilon \delta'\cdot
\]
which yields a contradiction as above. Hence, (4.5) holds. It follows that

$$s_{i,i+1} = R a_0 \frac{x_{i+1}}{x_i} \geq R \delta' a_0 \quad \text{for all } i \in \mathcal{A}_+.$$  

Therefore, by Lemmas 1.3 and 2.6 or their straightforward modifications if necessary, (4.6) shows that $S$ has the desired properties. One then concludes that the system $rx^T = x^T A$ has a unique solution up to constant multiples.

Now, our task becomes twofold. That is, (i) obtain the convergence norm $r$ of $A$ and (ii) solve the system $rx^T = x^T A$. Let $p_0(n)$ and $q_0(n)$ be as in (0.2) and (0.3), respectively. If $r$ in (i) is strictly less than 1 and the solution $x = (x_i)$ in (ii) has the convergent sum $x^T 1 < \infty$, then Theorem 3.1 guarantees that $q_0(n)$ converge to $q_i = x_i / x^T 1$ as $n \to \infty$. In the following, we show that if $m$ in (4.2) is negative then the quasi-stationary distribution exists. A transformed solution of $q = (q_i)$ is also derived.

First, we obtain the convergence norm $r$. To do so, as before, let

$$f_n = \text{Pr}[T(0) = n \mid X(0) = 1] \quad \text{for } n = 1, 2, \ldots,$$

and define $f(z) = \sum_{n=1}^{\infty} f_n z^n$. It should be noted that, since $f_n = a_0 u^T A^{n-1} u$ where $u = (1, 0, 0, \cdots)^T$, $f(z)$ has the same convergence radius as $P_0(z)$ in (1.1). Hence, $R = \sup \{z \geq 0 : f(z) < \infty\}$. On the other hand, due to the spatial homogeneity, it is not hard to see that

$$f(z) = z \sum_{k=0}^{\infty} a_k \{f(z)\}^k = z A(f(z)), \quad |z| < 1. \tag{4.7}$$

For each $\eta > 0$, consider then the equation

$$\eta y = \sum_{k=0}^{\infty} a_k y^k = A(y). \tag{4.8}$$

If solutions to (4.8) exist for $\eta = z^{-1}$, $f(z)$ is the minimum solution. For example, for $\eta = 1$, $f(z) = 1$ is the minimal solution if $m < 0$ whereas $f(z) < 1$ when $m > 0$.

Let $R_A$ be the convergence radius of $A(z)$. We first suppose that $A(R_A) = \infty$. Then, one can always draw a line $y = \gamma z$ tangent to $A(z)$. Note that $A(z)$ is strictly convex in $z \geq 0$ under the assumptions in (4.1). Hence, the tangent line is unique. Moreover, the slope $\gamma$ and the touching point $\theta$ can be determined by solving the simultaneous equations

$$\gamma \theta = A(\theta) \quad \text{and} \quad A'(\theta) = \gamma. \tag{4.9}$$

It is easy to see that

$$\begin{align*}
0 < \gamma, \theta < 1, & \quad \text{if and only if } A'(1) > 1, \\
\gamma = \theta = 1, & \quad \text{if and only if } A'(1) = 1, \\
0 < \gamma < 1, \theta > 1, & \quad \text{if and only if } A'(1) < 1.
\end{align*} \tag{4.10}$$
If \( A(R_A) < \infty \), we may not be able to draw such a line. The possibility, however, occurs only if \( A(R_A) \leq R_A \) and \( A'(R_A - ) < 1 \). If this is the case, we define \( \gamma \) and \( \theta \), instead of (4.9), by

\[
\text{(4.11)} \quad \gamma = A(R_A)/R_A \quad \text{and} \quad \theta = R_A,
\]

respectively.

Returning to the minimum solution to (4.8), if \( \eta < \gamma \) then \( y = \infty \) while \( y < \infty \) if \( \eta \geq \gamma \). To see this, consider a successive substitution \( \eta y_{n+1} = A(y_n) \) starting with \( y_0 = 0 \). It is easy to see that \( y_n \) is monotonically non-decreasing in \( n \). So, \( y_\infty = \lim_{n \to \infty} y_n \) exists, possibly infinite, and \( y_\infty \) satisfies (4.8). When \( \eta \geq \gamma \), one can show by induction, using (4.9) or (4.11) according to the possibility that \( y_n \leq \theta \) for all \( n \) so that \( y_\infty \leq \theta \). When \( \eta < \gamma \), \( y_\infty = \infty \). Hence, since \( f(z) \) is the minimum solution with \( z = \eta^{-1} \) so that \( y_\infty = f(z) \), the convergence norm of \( A \) is \( \gamma \) in (4.9) or (4.11) (see Kijima [10] for related results). Note that, since \( f(\gamma^{-1}) = \theta \), the substochastic matrix \( A \) is \( R \)-transient with \( R = \gamma^{-1} \). Thus, without using our results, one needs to prove the convergence of \( q_0(n) \) in (0.3) by some means. In fact, Daley [3] proved it by using Kemeny's ratio limit theorem for aperiodic random walks.

To solve the system \( \gamma x^T = x^T A \) as the second task, let \( x(z) = \sum_{n=1}^{\infty} x_n z^n \) with \( x = (x_i) \). A direct calculation then yields

\[
\text{(4.12)} \quad x(z) = \frac{x_1a_0z}{A(z) - \gamma z}.
\]

In order for \( \sum_{n=1}^{\infty} x_n < \infty \), \( x(z) \) must be regular at least in \( |z| < 1 \). Hence, it is required that \( A(z) - \gamma z > 0 \) in \( |z| < 1 \). This is so only if \( A'(1) < 1 \) or, equivalently, \( m < 0 \). For this case, in order for \( x(1) = 1 \), (4.12) yields

\[
\text{(4.13)} \quad x(z) = \frac{(1 - \gamma)z}{A(z) - \gamma z},
\]

which agrees with Theorem 1 of Daley [3].

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References


