ON THE CONVERGENCE TO STATIONARITY
OF BIRTH–DEATH PROCESSES

PAULINE COOLEN-SCHRIJNER,* University of Durham
ERIK A. VAN DOORN,** University of Twente

Abstract

Taking up a recent proposal by Stadje and Parthasarathy in the setting of the many-server Poisson queue, we consider the integral \( \int_0^\infty \lim_{u \to \infty} E(X(u)) - E(X(t)) \, dt \) as a measure of the speed of convergence towards stationarity of the process \( X(t), \ t \geq 0 \), and evaluate the integral explicitly in terms of the parameters of the process in the case that \( \{X(t), \ t \geq 0\} \) is an ergodic birth–death process on \( \{0, 1, \ldots\} \) starting in 0. We also discuss the discrete-time counterpart of this result, and examine some specific examples.

Keywords: Birth–death process; speed of convergence

AMS 2000 Subject Classification: Primary 60J80

1. Introduction

Let \( X(t) \) be the number of customers at time \( t \) in a stable \( M/M/c \) queueing system and suppose that the system is initially empty. The process \( \{X(t), \ t \geq 0\} \) is then stochastically increasing, and, as a consequence, \( E(X(t)) \) converges monotonically to its limiting value \( M := \lim_{t \to \infty} E(X(t)) \). This has recently motivated Stadje and Parthasarathy [10] to propose the quantity

\[
\int_0^\infty [M - E(X(t))] \, dt
\]

as a measure of the speed of convergence as \( t \to \infty \) of the distribution of \( X(t) \) to the stationary distribution of the number of customers in an \( M/M/c \) system. They subsequently evaluate the integral (1.1) explicitly in terms of the number of servers \( c \), and the arrival and service rates of the system.

Clearly, the process \( \{X(t), \ t \geq 0\} \) constitutes a birth–death process. Moreover, any birth–death process on the nonnegative integers which starts in state 0 is stochastically increasing (see, for example, [9, Section 4.8]). It is therefore natural to ask whether the result of Stadje and Parthasarathy can be extended into the more general setting of birth–death processes. The purpose of this paper is to resolve this question in the affirmative. So in what follows \( X := \{X(t), \ t \geq 0\} \) will be an ergodic birth–death process taking values in \( \mathcal{N} := \{0, 1, \ldots\} \) with birth rates \{\( \lambda_j, \ j \in \mathcal{N} \)\} and death rates \{\( \mu_j, \ j \in \mathcal{N} \)\}, all strictly positive except \( \mu_0 = 0 \). Throughout we will assume that \( X(0) = 0 \) and use the notation

\[
p_j(t) := \Pr\{X(t) = j \mid X(0) = 0\}, \quad j \in \mathcal{N}, \ t \geq 0,
\]

and \( p_j := \lim_{t \to \infty} p_j(t), \ j \in \mathcal{N} \).
The speed of convergence to stationarity of the process \( \mathcal{X} \) is usually characterized by the quantity

\[
\alpha(\mathcal{X}) := \sup\{\alpha \geq 0 \mid p_j - p_j(t) = O(\exp(-\alpha t)) \text{ as } t \to \infty \text{ for all } j \in \mathcal{N}\},
\]
or its reciprocal \( r(\mathcal{X}) := 1/\alpha(\mathcal{X}) \), the relaxation time of \( \mathcal{X} \) (see, for example, [1] and [12]). If

\[
M := \lim_{t \to \infty} E(X(t)) < \infty \text{ we also have}
\]

\[
r(\mathcal{X}) = \inf\{r > 0 \mid M - E(X(t)) = O(\exp(-t/r)) \text{ as } t \to \infty\},
\]

the infimum of an empty set being infinity. The relaxation times of many specific birth–death processes are known, but there exists no general expression for \( r(\mathcal{X}) \) in terms of the birth and death rates of \( \mathcal{X} \). Since, as we will show, the integral (1.1) can be evaluated explicitly in terms of the birth and death rates of \( \mathcal{X} \) it may be an attractive alternative to \( r(\mathcal{X}) \) as a one-parameter characterization of the speed of convergence. Rather than (1.1), however, we propose its normalized value

\[
m(\mathcal{X}) := \int_0^\infty \left[1 - E(X(t))/M\right] dt
\]
as an alternative to \( r(\mathcal{X}) \) as a measure of the speed of convergence towards stationarity of the process \( \mathcal{X} \).

The rest of the paper is organized as follows. After presenting some preliminary results on birth–death processes in Section 2, we will obtain our main result—an explicit expression for the integral (1.1) in terms of the birth and death rates—in Section 3. The expression will be evaluated for some specific birth–death processes in Section 4. Finally, in Section 5, we consider birth–death processes in discrete time, and show that a similar result may be obtained in this setting by performing a suitable transformation, provided the birth and death probabilities satisfy certain requirements.

2. Preliminaries

The potential coefficients of the birth–death process \( \mathcal{X} := \{X(t), \ t \geq 0\} \) are defined by

\[
\pi_0 := 1 \quad \text{and} \quad \pi_j := \lambda_0 \lambda_1 \cdots \lambda_{j-1} / \mu_1 \mu_2 \cdots \mu_j, \quad j \geq 1.
\]

Since \( \mathcal{X} \) is assumed to be ergodic these constants must satisfy the condition

\[
K := \sum_{j=0}^\infty \pi_j < \infty.
\]

We will additionally assume that

\[
\sum_{j=0}^\infty (\lambda_j \pi_j)^{-1} = \infty,
\]

ensuring that \( \mathcal{X} \) is uniquely determined by its birth and death rates (see [5]).

It is well known that

\[
p_j := \lim_{t \to \infty} p_j(t) = \frac{\pi_j}{K}, \quad j \in \mathcal{N},
\]
while (see, for example, [4])
\[
\lim_{t \to \infty} E(X(t)) = E(X),
\]
with \(X\) denoting a random variable with distribution \([p_j, \ j \in \mathcal{N}]\). Evidently, we will assume throughout that
\[
E(X) = \sum_{j=0}^{\infty} j p_j < \infty. \tag{2.4}
\]

It will be convenient to introduce the quantities
\[
\tau_j := p_j \sum_{k=0}^{j-1} (\lambda_k p_k)^{-1} \sum_{\ell=k+1}^{\infty} p_\ell, \quad j \geq 0, \tag{2.5}
\]
and \(T := \sum_{j=0}^{\infty} \tau_j\). Here, and henceforth, the empty sum should be interpreted as 0 (so that \(\tau_0 := 0\)). By interchanging summations it is easily seen that
\[
T = \sum_{k=0}^{\infty} (\lambda_k p_k)^{-1} \left( \sum_{\ell=k+1}^{\infty} p_\ell \right)^2,
\]
which may be finite or infinite. It is shown in [2] that \(T\) may be interpreted as the mean first entrance time to 0 from the ergodic distribution.

From Karlin and McGregor [5] we obtain a representation for the time-dependent probabilities \(p_j(t)\). It involves a sequence of polynomials \(\{Q_j(x), \ j \geq 0\}\), which are determined uniquely by the recurrence relation
\[
-x Q_j(x) = \mu_j Q_{j-1}(x) - (\lambda_j + \mu_j) Q_j(x) + \lambda_j Q_{j+1}(x), \quad j \geq 0,
\]
together with \(Q_{-1}(x) = 0\) and \(Q_0(x) = 1\), and a positive Borel measure \(\psi\) with total mass 1 and with infinite support on the nonnegative real axis. Namely, we have
\[
p_j(t) = \pi_j \int_0^\infty e^{-xt} Q_j(x) \psi(dx), \quad j \in \mathcal{N}, \ t \geq 0. \tag{2.6}
\]
The polynomials \(\{Q_j(x), \ j \geq 0\}\) are the birth–death polynomials associated with \(X\), and are orthogonal with respect to \(\psi\), namely,
\[
\pi_j \int_0^\infty Q_i(x) Q_j(x) \psi(dx) = \delta_{ij}, \quad i, j \geq 0,
\]
where \(\delta_{ij}\) denotes the Kronecker delta. The measure \(\psi\) is called the spectral measure of \(X\). In view of (2.3) and (2.6) the measure \(\psi\) must have an atom at 0 of size
\[
\psi([0]) = \frac{1}{K},
\]
and we can write
\[
p_j(t) - p_j = \pi_j \int_{0^+}^\infty e^{-xt} Q_j(x) \psi(dx), \quad j \in \mathcal{N}, \ t \geq 0. \tag{2.7}
\]
We shall have use for the dual process $X^d := \{X^d(t), \ t \geq 0\}$ associated with $X$, which (following [6]) we define to be the birth–death process on $\mathcal{N}$ with birth rates $\{\lambda_j, \ j \geq 0\}$ and death rates $\{\mu_j, \ j \geq 0\}$, given by

$$
\lambda_j^d := \mu_{j+1} \quad \text{and} \quad \mu_j^d := \lambda_j, \quad j \in \mathcal{N}.
$$

Note that in $X^d$, probability mass can escape from $\mathcal{N}$, via state 0, into an ignored absorbing state $-1$, say, since $\mu_0^d = \lambda_0 > 0$. The potential coefficients of the dual process are easily seen to satisfy

$$
\pi_j^d = \lambda_0 (\lambda_j \pi_j)^{-1} \quad \text{and} \quad \mu_0^d (\lambda_j^d \pi_j^d)^{-1} = \pi_{j+1}, \quad j \geq 0,
$$

so that by (2.1) and (2.2) we have

$$
\sum_{j=0}^{\infty} \pi_j^d = \infty \quad \text{and} \quad \sum_{j=0}^{\infty} (\lambda_j^d \pi_j^d)^{-1} < \infty.
$$

Denoting by $\{Q_j^d(x), \ j \geq 0\}$ the birth–death polynomials corresponding to the dual process and letting $Q_{-1}^d(x) = 0$, we also have

$$
\pi_j Q_j(x) = Q_j^d(x) - Q_{j-1}^d(x), \quad j \geq 0,
$$

and

$$
Q_j^d(0) = 1 + \mu_0^d \sum_{k=0}^{j-1} (\lambda_k^d \pi_k^d)^{-1} = \sum_{k=0}^{j} \pi_k, \quad j \geq 0.
$$

Finally, the spectral measures $\psi$ and $\psi^d$ of $X$ and $X^d$, respectively, are related as

$$
\mu_0^d \psi^d([0, x]) = \int_0^x u \psi(du), \quad x \geq 0,
$$

so that in particular

$$
\psi^d([0]) = 0.
$$

The above relations between $X$ and $X^d$ can be found in [6], [12] or [9].

### 3. The main result

In this section we will first evaluate the integrals

$$
I_j := \int_0^\infty [p_j(t) - p_j] \, dt, \quad j \geq 0,
$$

after which the value of the integral (1.1) will follow as a corollary. Since $p_j(t)$ is a unimodal function (see [8]) the integrals $I_j$ exist, but may be infinite.

The integrals $I_j$ have been evaluated explicitly by Whitt [13, Proposition 6] in the setting of a birth–death process with finite state space $\{0, 1, \ldots, n\}$. To obtain them in the present setting we could let $n$ tend to infinity in the expressions given by Whitt and justify the implicit interchange of limit and integration, the latter being possible on the basis of results in [2]. However, we will evaluate $I_j$ directly in the infinite setting at hand, by using Karlin and McGregor’s spectral representation for $p_j(t)$. Indeed, substituting (2.7) into (3.1) and interchanging the integrals
(which is allowed by Fubini’s theorem since the integrand has only finitely many sign changes at fixed values of \(x\)) gives us

\[
I_j = \int_{0^+}^{\infty} \pi_j Q_j(x)x^{-1} \psi(dx), \quad j \geq 0.
\] (3.2)

Integrals of the type

\[
\int_{0}^{\infty} Q_j(x)x^{-k} \psi(dx), \quad j \geq 0, \quad k \geq 1.
\]

have been evaluated explicitly by Karlin and McGregor [6] under conditions, however, which do not correspond to the present setting (in which there is an atom at 0). We can nevertheless exploit these results—as suggested by Karlin and McGregor themselves—by translating the integral (3.2) in terms of the dual process \(\mathcal{X}^d\). Pursuing this approach we substitute (2.10) and (2.12) into (3.2) and use (2.13) to conclude that the integrals (3.2) can be represented as

\[
I_j = \mu_0^d \int_{0}^{\infty} [Q_j^d(x) - Q_{j-1}^d(x)]x^{-2} \psi^d(dx), \quad j \geq 0.
\] (3.3)

Since the dual process satisfies the condition (2.9), we can subsequently evoke the results in [6, Section 9B] to evaluate the integrals

\[
H_j := \mu_0^d \int_{0}^{\infty} Q_j^d(x)x^{-2} \psi^d(dx), \quad j \geq 0.
\] (3.4)

Namely, exploiting [6, equation (9.16)] and the duality relations (2.8) and (2.11) we readily see that the integrals \(H_j\) satisfy

\[
H_j = T \sum_{k=0}^{j} p_k - \sum_{k=0}^{j} r_k, \quad j \geq 0,
\]

which should be interpreted as \(\infty\) if \(T = \infty\).

Since, by (2.11), we have \(Q_j^d(0) - Q_{j-1}^d(0) = \pi_j > 0\), it is clear from (3.3) and (3.4) that \(I_j\) is finite if and only if \(H_0\) is finite, that is, \(T < \infty\), in which case \(H_0 = H_0\) and \(I_j = H_j - H_{j-1}, j \geq 1\). Hence we obtain the next theorem, which gives the same result obtained by letting \(n\) tend to infinity in the expression for \(I_j\) given by Whitt [13, Proposition 6] in a finite setting.

**Theorem 3.1.** We have

\[
I_j := \int_{0}^{\infty} [p_j(t) - p_j] dt = T p_j - \tau_j, \quad j \geq 0.
\] (3.5)

which should be interpreted as \(\infty\) if \(T = \infty\).

**Theorem 3.2.** If \(\sum_{j=0}^{\infty} j \tau_j < \infty\), then

\[
\int_{0}^{\infty} [E(X) - E(X(t))] dt = \sum_{j=0}^{\infty} j \tau_j - T E(X),
\]

whereas the integral is infinite otherwise.
Proof. Since $X$ is stochastically increasing, we have
\[ \sum_{j=0}^{k} (p_j(t) - p_j) > 0, \quad k \geq 0. \] (3.7)
We also observe that
\[ E(X) - E(X(t)) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} (p_j - p_j(t)) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} (p_j(t) - p_j). \] (3.8)
It follows that $E(X) - E(X(t)) > p_0(t) - p_0$, and hence, by (3.5), the integral is infinite if $T = \infty$. Now assuming that $T < \infty$, and using (3.8) and the fact that $\sum I_j = 0$, we can write
\[ \int_{0}^{\infty} [E(X) - E(X(t))] \, dt = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} I_j = -\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} I_j = -\sum_{j=1}^{\infty} j I_j, \]
the interchange of integration and summation being justified by (3.7). In view of (2.4) and (3.5) the theorem follows.

4. Examples

To check the theorem we first look at a process for which the value of the integral (1.1) is available. Namely, we let $X := \{X(t), t \geq 0\}$ be the number of customers in the $M/M/\infty$ queue, which is a birth–death process with rates
\[ \lambda_j = \lambda \quad \text{and} \quad \mu_j = j \mu, \quad j \in \mathcal{N}. \]
It is well known (see, for example, [3, p. 461]) that when the system starts empty the mean number of customers in the system at time $t$ is given by
\[ E(X(t)) = \frac{\lambda}{\mu} (1 - e^{-\mu t}), \quad t \geq 0, \]
so that
\[ \int_{0}^{\infty} [E(X) - E(X(t))] \, dt = \frac{\lambda}{\mu^2}. \]
This result can indeed be recovered—albeit somewhat tediously—by evaluating the right-hand side of (3.6). For completeness’ sake we note that the convergence measures (1.2) and (1.3) for this process are given by
\[ m(X) = r(X) = \frac{1}{\mu}. \]
Our second example is the birth–death process $X$ with rates
\[ \lambda_j = \frac{\lambda}{j+1} \quad \text{and} \quad \mu_{j+1} = \mu, \quad j \geq 0, \]
which may be interpreted as the process of the number of customers in a queueing system in which customers are discouraged by queue length (see, for example, [11]). In this case no simple expression for $E(X(t))$ is available. To evaluate the right-hand side of (3.6) we write
\[ a := \frac{\lambda}{\mu} \] (4.1)
and note that \( K = e^a \) and \( E(X) = a \). Moreover, letting

\[
f_j(a) := \sum_{\ell=1}^{\infty} \frac{(j + 1)!}{(j + \ell)!} a^\ell, \quad j \geq 0, \tag{4.2}
\]

we readily obtain

\[
\tau_j = \frac{1}{\lambda} e^{-a} \frac{a^j}{j!} \sum_{k=0}^{j-1} f_k(a), \quad j \geq 0,
\]

so that

\[
T = \frac{1}{\lambda} e^{-a} \sum_{j=1}^{\infty} \frac{a^j}{j!} \sum_{k=0}^{j-1} f_k(a) \quad \text{and} \quad \sum_{j=1}^{\infty} j \tau_j = a^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} \sum_{k=0}^{j} f_k(a).
\]

Substitution of these results in (3.6) gives us

\[
\int_0^\infty [E(X) - E(X(t))] \, dt = \frac{a^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} f_j(a)}{\lambda}.
\]

which, after substitution of (4.1) and (4.2) and some algebra, reduces to

\[
\int_0^\infty [E(X) - E(X(t))] \, dt = \frac{\lambda(\lambda + 2\mu)}{2\mu^3}.
\]

It now follows that

\[
m(X) = \frac{\lambda + 2\mu}{2\mu^2},
\]

while we know from [11] that the relaxation time of the process is given by

\[
r(X) = \frac{\lambda + 2\mu + \sqrt{\lambda^2 + 4\lambda\mu}}{2\mu^2}.
\]

We will finally apply our results to the process of the number of customers in an \( M/M/c \) queueing system—the setting in which Stadje and Parthasarathy [10] proposed the integral (1.1) as a measure of the speed of convergence to stationarity—and compare our findings with those in [10]. The process at hand is a birth–death process \( X \) with rates

\[
\lambda_j = \lambda \quad \text{and} \quad \mu_j = \min\{j, c\} \mu, \quad j \in \mathcal{N}.
\]

Writing

\[
\rho := \frac{\lambda}{c\mu}, \tag{4.3}
\]

we must have \( \rho < 1 \) for the system to be stable. The potential coefficients of the process are given by

\[
\pi_j = \begin{cases} 
\frac{(cp)^j}{j!}, & 0 \leq j \leq c, \\
\frac{e^c \rho^j}{c!}, & j \geq c,
\end{cases}
\]
so, with
\[ K_c := \sum_{j=c}^{\infty} \pi_j = \frac{1}{1 - \rho} \frac{(cp)^c}{c!}, \quad (4.4) \]
we have
\[ K = \sum_{j=0}^{c-1} \frac{(cp)^j}{j!} + K_c \quad \text{and} \quad E(X) = cp + \frac{\rho}{1 - \rho} \frac{K_c}{K}. \quad (4.5) \]

It is convenient to let
\[ A_j := \sum_{k=0}^{j-1} (\lambda_k p_k)^{-1} \sum_{\ell=k+1}^{\infty} p_\ell, \quad j \geq 0 \]
(so that \( A_0 := 0 \)), which is readily seen to imply that
\[ A_j = \frac{1}{\lambda} \sum_{k=0}^{j-1} \frac{k!}{(cp)^k} \left( \sum_{\ell=k+1}^{c-1} \frac{(cp)^\ell}{\ell!} + K_c \right), \quad 0 \leq j \leq c. \quad (4.6) \]
The quantities \( \tau_j \) of (2.5) can now be expressed as
\[ \tau_j = \begin{cases} \frac{A_j (cp)^j}{K} \frac{1}{j!}, & 0 \leq j \leq c, \\ \frac{1}{K} \left( A_c + (j - c) - \rho \frac{1}{1 - \rho} \right)^j \frac{c^c \rho^j}{c!}, & j \geq c, \end{cases} \]
from which it follows after some algebra that
\[ T = \frac{1}{K} \sum_{j=1}^{c-1} \frac{A_j (cp)^j}{j!} + T_c \quad (4.7) \]
and
\[ \sum_{j=0}^{\infty} j \tau_j = \frac{cp}{K} \sum_{j=0}^{c-2} \frac{A_{j+1} (cp)^j}{j!} + T_c \left( c + \frac{\rho}{1 - \rho} \right) + \frac{K_c}{K} \frac{1}{\lambda} \frac{\rho^2}{(1 - \rho)^3}, \quad (4.8) \]
where
\[ T_c := \sum_{j=c}^{\infty} \tau_j = \frac{K_c}{K} \left( A_c + \frac{1}{\lambda} \frac{\rho^2}{(1 - \rho)^3} \right). \quad (4.9) \]
The integral (1.1) can now easily be evaluated for specific values of \( c, \lambda \) and \( \mu \) from (3.6) and the expressions (4.3)–(4.9). In particular, for \( c = 1 \) we obtain
\[ \int_0^{\infty} [E(X) - E(X(t))] \, dt = \frac{1}{\mu} \frac{\rho}{(1 - \rho)^3}. \]
As a consequence the measure (1.3) for the \( M/M/1 \) queue is given by
\[ m(\mathcal{X}) = \frac{1}{\mu} \frac{1}{(1 - \rho)^2}, \]
while it is well known that the relaxation time of the $M/M/1$ queue satisfies

$$r(\mathcal{X}) = \frac{1}{\mu} \frac{(1 + \sqrt{\rho})^2}{(1 - \rho)^2}.$$ 

Evaluating (3.5) for $c = 2$ leads to

$$\int_0^\infty [E(X) - E(X(t))] \, dt = \frac{1}{\mu} \frac{2\rho(1 - \rho + \rho^2)}{(1 - \rho)^3(1 + \rho)^2},$$

so in this case we have

$$m(\mathcal{X}) = \frac{1}{\mu} \frac{1 - \rho + \rho^2}{(1 - \rho)^2(1 + \rho)},$$

while the relaxation time of the $M/M/2$ queue is given in [1] as

$$r(\mathcal{X}) = \begin{cases} \frac{1}{\mu} \frac{2}{1 + 4\rho + \sqrt{1 - 8\rho}}, & 0 < \rho < \frac{1}{9}, \\ \frac{1}{2\mu} \frac{(1 + \sqrt{\rho})^2}{(1 - \rho)^2}, & \frac{1}{9} \leq \rho < 1. \end{cases}$$

Stadje and Parthasarathy have informed the authors that there is a computational error in the expression for $B_1$ in Theorem 3 of their paper [10], which should read

$$B_1 = \pi_{c-1} \left[ (c - 1) f_{c-2}(0) - \frac{c\lambda}{(c - \lambda)^3} \right].$$

With this correction our results for $c = 1$ and $c = 2$ are in agreement with those of Stadje and Parthasarathy [10].

5. Discrete-time birth–death processes

A discrete-time birth–death process or random walk $\tilde{\mathcal{X}} := \{\tilde{X}(n), \ n = 0, 1, \ldots \}$ on the state space $\mathcal{N} := \{0, 1, \ldots \}$ is a Markov chain with stationary one-step transition probabilities $p_{ij}$ satisfying $p_{ij} = 0$ for $|i - j| > 1$. We shall only consider honest random walks in which $p_j := p_{j,j+1} > 0$, $q_{j+1} := p_{j+1,j} > 0$, and $r_j := p_{jj} \geq 0$ for all $j \in \mathcal{N}$, but $r_j > 0$ for at least one $j \in \mathcal{N}$ (the latter to avoid periodicity). We assume throughout that $\tilde{X}(0) = 0$ and let

$$\tilde{p}_j(n) := \Pr(\tilde{X}(n) = j \mid \tilde{X}(0) = 0), \quad j \in \mathcal{N}, \ n \geq 0.$$ 

Defining

$$\tilde{\pi}_0 = 1 \quad \text{and} \quad \tilde{\pi}_j = \frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}, \quad j \geq 1,$$

it is well known that the process is ergodic if

$$\tilde{K} := \sum_{j=0}^\infty \tilde{\pi}_j < \infty,$$

in which case

$$\tilde{p}_j = \lim_{n \to \infty} \tilde{p}_j(n) = \frac{\tilde{\pi}_j}{\tilde{K}}, \quad j \in \mathcal{N}.$$
and

\[
\lim_{n \to \infty} E(\tilde{X}(n)) = E(\tilde{X}) = \sum_{j=0}^{\infty} j \tilde{p}_j,
\]

with \( \tilde{X} \) denoting a random variable with distribution \( \{ \tilde{p}_j, j \in \mathcal{N} \} \) (see, for example, [7]).

If \( E(\tilde{X}) \) is finite it seems natural to propose—by analogy with (1.1)—the sum

\[
\sum_{n=0}^{\infty} [E(\tilde{X}) - E(\tilde{X}(n))]
\]

as a measure of the speed of convergence of \( \tilde{X}(n) \) to \( \tilde{X} \), provided \( E(\tilde{X}(n)) \) converges monotonically to \( E(\tilde{X}) \). However, it is easy to construct examples of random walks starting in 0 in which the latter does not happen, so that (5.1) is less attractive than its continuous-time counterpart as a measure of the speed of convergence to stationarity. For completeness' sake we shall nevertheless evaluate the sum (5.1) explicitly, under the condition that

\[
E(\tilde{X}(n)) < E(\tilde{X}), \quad n \geq 0.
\]

We note that a sufficient condition for \( E(\tilde{X}(n)) \) to converge monotonically to its limit \( E(\tilde{X}) \) as \( n \to \infty \) (and hence for (5.2)), is stochastic monotonicity of \( \tilde{X} \), which prevails if and only if

\[
p_j + q_{j+1} < 1, \quad j \in \mathcal{N}
\]

(see [9, Example 3.12]).

To evaluate the sum (5.1) we associate with \( \tilde{X} \) a continuous-time birth–death process \( X := \{ X(t), t \geq 0 \} \) with rates

\[
\lambda_j = p_j \quad \text{and} \quad \mu_j = q_j, \quad j \in \mathcal{N}.
\]

Since \( \lambda_j + \mu_j = p_j + q_j \leq 1 \) for all \( j \), the process \( X \) is uniformizable with uniformization parameter 1 and we get \( \tilde{X} \) back as the uniformized process. Moreover, with \( \{ N(t), t \geq 0 \} \) denoting a Poisson process with intensity 1, we have

\[
\{ X(t), t \geq 0 \} \overset{d}{=} \{ \tilde{X}(N(t)), t \geq 0 \}
\]

(see, for example, [9, Section 4.4] for these results on uniformization). The next theorem shows that the problem of evaluating (5.1) can now be reduced to that of evaluating the integral (1.1) for the continuous-time process \( X \).

**Theorem 5.1.** If \( E(\tilde{X}(n)) < E(\tilde{X}) \) for all \( n \geq 0 \), then

\[
\sum_{n=0}^{\infty} [E(\tilde{X}) - E(\tilde{X}(n))] = \int_{0}^{\infty} [E(X) - E(X(t))] \, dt,
\]

where \( \{ X(t), t \geq 0 \} \) is the birth–death process with rates (5.3).
Proof. It is obvious from (5.4) that $E(\tilde{X}) = E(X)$. Moreover, by conditioning on the value of $N(t)$ we get

$$
\int_0^\infty [E(X) - E(X(t))] \, dt = \int_0^\infty [E(\tilde{X}) - E(\tilde{X}(N(t)))] \, dt
$$

$$
= \int_0^\infty \left\{ \sum_{n=0}^\infty [E(\tilde{X}) - E(\tilde{X}(n))] e^{-t} \frac{t^n}{n!} \right\} \, dt
$$

$$
= \sum_{n=0}^\infty [E(\tilde{X}) - E(\tilde{X}(n))],
$$

where the interchange of integration and summation is allowed by Fubini's theorem.

Acknowledgement

This work was partly done while the second author was visiting the University of Durham.

References