FAMILIES OF BIRTH–DEATH PROCESSES WITH
SIMILAR TIME-DEPENDENT BEHAVIOUR

R. B. LENIN, * University of Antwerp
P. R. PARTHASARATHY,** Indian Institute of Technology
W. R. W. SCHEINHARDT,*** Eindhoven University of Technology
E. A. VAN DOORN,**** University of Twente

Abstract

We consider birth–death processes taking values in \( \mathcal{N} \equiv \{0, 1, \ldots \} \), but allow the death rate in state 0 to be positive, so that escape from \( \mathcal{N} \) is possible. Two such processes with transition functions \( \{p_{ij}(t)\} \) and \( \{\tilde{p}_{ij}(t)\} \) are said to be similar if, for all \( i, j \in \mathcal{N} \), there are constants \( c_{ij} \) such that \( \tilde{p}_{ij}(t) = c_{ij}p_{ij}(t) \) for all \( t \geq 0 \). We determine conditions on the birth and death rates of a birth–death process for the process to be a member of a family of similar processes, and we identify the members of such a family. These issues are also resolved in the more general setting in which the two processes are called similar if there are constants \( c_{ij} \) and \( v \) such that \( \tilde{p}_{ij}(t) = c_{ij}e^{vt}p_{ij}(t) \) for all \( t \geq 0 \).

**Keywords:** Absorption probability; chain sequence; invariant vector; transient behaviour; transition function

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1. Introduction

In this paper a birth–death process \( \mathcal{X} \equiv \{X(t), t \geq 0\} \), say, will always be a process taking values in \( \mathcal{N} \equiv \{0, 1, \ldots \} \) with birth rates \( \{\lambda_n, n \in \mathcal{N}\} \) and death rates \( \{\mu_n, n \in \mathcal{N}\} \), all strictly positive except \( \mu_0 \), which might be equal to 0. When \( \mu_0 = 0 \) the process is irreducible, but when \( \mu_0 > 0 \) the process may escape from \( \mathcal{N} \), via 0, to an absorbing state \(-1\).

The \( q \)-matrix of transition rates of \( \mathcal{X} \), restricted to the states in \( \mathcal{N} \), will be denoted by 
\[
Q = \begin{pmatrix}
-(\lambda_0 + \mu_0) & \lambda_0 & 0 & 0 & 0 & \cdots \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \cdots \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
& & & & & \\
& & & & & 
\end{pmatrix}.
\]

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* Postal address: Department of Mathematics and Computer Science, University of Antwerp, Universiteitsplein 1, B-2610 Wilrijk, Belgium.

** Postal address: Department of Mathematics, Indian Institute of Technology, Madras, Chennai 600 036, India.

*** Postal address: Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

**** Postal address: Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. Email address: doorn@math.utwente.nl
We shall assume that the transition functions
\[ p_{ij}(t) \equiv \Pr\{X(t) = j \mid X(0) = i\}, \quad i, j \in \mathcal{N}, \quad t \geq 0, \]
constitute the unique Q-function (i.e., set of transition functions having Q as q-matrix) satisfying both the Kolmogorov backward equations
\[ p'_{ij}(t) = \sum_{k \in \mathcal{N}} q_{ik} p_{kj}(t), \quad i, j \in \mathcal{N}, \quad t \geq 0, \tag{2} \]
and forward equations
\[ p'_{ij}(t) = \sum_{k \in \mathcal{N}} p_{ik}(t) q_{kj}, \quad i, j \in \mathcal{N}, \quad t \geq 0. \tag{3} \]
Equivalently, see [2, pp. 262–263], we assume that the potential coefficients
\[ \pi_0 \equiv 1 \quad \text{and} \quad \pi_n \equiv \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n = 1, 2, \ldots, \]
satisfy the condition
\[ \sum_{n=0}^{\infty} (\pi_n + (\lambda_n \pi_n)^{-1}) = \infty. \tag{4} \]

The problem of solving the Kolmogorov equations explicitly for a specific set of birth and death rates has been approached in the literature in many different ways (of which the method of Karlin and McGregor [8] involving orthogonal polynomials has probably been the most successful). Our approach to the problem of finding solutions to the Kolmogorov equations will be to investigate whether a known solution to the equations for one birth–death process can help us identify the transition functions of other birth–death processes, by establishing whether the given process belongs to a class of birth–death processes whose transition functions behave—in a sense to be defined—'similarly'.

Thus consider, besides \( X \), another birth–death process \( \tilde{X} \), determined by birth rates \( \tilde{\lambda}_n, \quad n \in \mathcal{N} \) and death rates \( \tilde{\mu}_n, \quad n \in \mathcal{N} \), with potential coefficients \( \tilde{\pi}_n \) and transition functions \( \tilde{p}_{ij}(t) \).

**Definition 1.** The birth–death processes \( X \) and \( \tilde{X} \) are said to be similar if the transition functions of \( X \) and \( \tilde{X} \) differ only by a constant factor, that is, if there are constants \( c_{ij}, \quad i, j \in \mathcal{N}, \) such that
\[ \tilde{p}_{ij}(t) = c_{ij} p_{ij}(t), \quad i, j \in \mathcal{N}, \quad t \geq 0. \tag{5} \]

This concept of similarity is seemingly more general than that of Di Crescenzo [6], who calls the birth–death processes \( X \) and \( \tilde{X} \) similar if there are constants \( v_n, \quad n \in \mathcal{N}, \) such that
\[ \tilde{p}_{ij}(t) = \frac{v_j}{v_i} p_{ij}(t), \quad i, j \in \mathcal{N}, \quad t \geq 0. \tag{6} \]

However, the next theorem shows that the two definitions actually amount to the same thing, and that birth and death rates and transition functions of two similar birth–death processes must be related in a very specific way.
Theorem 1. If $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are similar birth–death processes, then their birth and death rates are related as
\[ \tilde{\lambda}_n + \tilde{\mu}_n = \lambda_n + \mu_n, \quad \tilde{\lambda}_n \tilde{\mu}_{n+1} = \lambda_n \mu_{n+1}, \quad n \in \mathcal{N}, \] (7)
while their transition functions satisfy
\[ \tilde{p}_{ij}(t) = \sqrt{\frac{\pi_i \pi_j}{\pi_i \pi_j}} p_{ij}(t), \quad i, j \in \mathcal{N}, \quad t \geq 0. \] (8)

Since this theorem is a special case of Theorem 6 in Section 6, we do not give a separate proof.

One might suspect that, conversely, a birth–death process $\widetilde{\mathcal{X}}$ is similar to a given birth–death process $\mathcal{X}$ if the birth and death rates of $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ are related as in (7), but this is probably not true in general unless one imposes an additional condition, as in the next theorem.

Theorem 2. Let $\mathcal{X}$ be a birth–death process satisfying
\[ \sum_{n=1}^{\infty} \pi_n \left( \sum_{k=0}^{n-1} (\lambda_k \pi_k)^{-1} \right)^2 = \infty \] (9)
if $\mu_0 > 0$, and let $\widetilde{\mathcal{X}}$ be a birth–death process the rates of which are related to those of $\mathcal{X}$ as in (7). Then the processes $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are similar.

The proof of this theorem has been relegated to the Appendix. In what follows we will assume the validity of condition (9) (which is stronger than (4)) when $\mu_0 > 0$.

The questions to be answered now are under which conditions the phenomenon of similarity occurs and, if it occurs, whether one can identify all the birth–death processes which are similar to a given process. These problems were partly resolved by Di Crescenzo [6] who restricted his analysis to the case $\mu_0 > 0$ (see [7] for related results on bilateral birth–death processes). In Section 2 we obtain a complete solution, in which the crucial step is the application of a result of Chihara [5] on chain sequences.

In Section 3 we will show that our similarity concept is closely related to a duality concept for $Q$-functions introduced in connection with invariant vectors (see [17]). Some examples of families of similar birth–death processes are given in Section 4. In the first example our starting point will be the process with constant birth and death rates, while in the second example we consider the process with birth rates $\lambda_n \equiv n + a$, for some $a > 0$, and death rates $\mu_n \equiv n$, $n \in \mathcal{N}$. Section 5 contains some remarks on similarity for birth–death processes on a finite state space.

In Section 6 we discuss the following generalization of our similarity concept.

Definition 2. The birth–death process $\widetilde{\mathcal{X}}$ is said to be $\nu$-similar to the birth–death process $\mathcal{X}$ for some real number $\nu$ if there are constants $c_{ij}$, $i, j \in \mathcal{N}$, such that
\[ \tilde{p}_{ij}(t) = c_{ij} e^{\nu t} p_{ij}(t), \quad i, j \in \mathcal{N}, \quad t \geq 0. \] (10)

So similarity in the sense of Definition 1 is equivalent to 0-similarity. We will show that the problem of finding all birth–death processes $\widetilde{\mathcal{X}}$ which are $\nu$-similar to $\mathcal{X}$ for some real $\nu$ can easily be resolved by combining the results of Section 2 and a result which is closely related to a result of Küchler [11] on exponential families of birth–death processes.
The problem of identifying birth–death processes which are ν-similar to a given process has been addressed previously by Letessier and Valent (see [13], [14]) in the more restricted setting in which the parameters λ_n and μ_n are polynomials in n. Our Theorem 1 and its generalization, Theorem 6 in Section 6, can be partly traced back to these references.

2. Families of similar birth–death processes

We will assume that the transition functions \{p_{ij}(t), i, j \in \mathcal{N}\} of the birth–death process \(X \equiv \{X(t), t \geq 0\}\), with birth rates \{λ_n, n \in \mathcal{N}\} and death rates \{μ_n, n \in \mathcal{N}\} satisfying (4) and, if \(μ_0 > 0\), (9), are known. We let

\[
α_n \equiv λ_n + μ_n, \quad β_{n+1} \equiv λ_n μ_{n+1}, \quad n \in \mathcal{N}.
\]  

(11)

In view of Theorem 1, the question of which birth–death processes are similar to \(X\) may now be phrased as follows. Can we identify, besides \{λ_n, n \in \mathcal{N}\} and \{μ_n, n \in \mathcal{N}\}, all other sets of birth rates \{\tilde{λ}_n, n \in \mathcal{N}\} and death rates \{\tilde{μ}_n, n \in \mathcal{N}\} such that

\[
\tilde{λ}_n + \tilde{μ}_n = α_n, \quad \tilde{λ}_n \tilde{μ}_{n+1} = β_{n+1}, \quad n \in \mathcal{N}.
\]  

(12)

This problem can be transformed into a problem involving chain sequences. Let us recall the definition and some basic results; see [3, Section III.5] and [5] for proofs and developments.

A sequence \(\{a_n\}_{n=1}^\infty\) is a chain sequence if there exists a second sequence \(\{g_n\}_{n=0}^\infty\) such that

\[
\begin{align*}
(i) \quad 0 &\leq g_0 < 1, \quad 0 < g_n < 1, \quad n = 1, 2, \ldots, \\
(ii) \quad a_n &= (1 - g_{n-1})g_n, \quad n = 1, 2, \ldots.
\end{align*}
\]

(13)

The sequence \(\{g_n\}\) is called a parameter sequence for \(\{a_n\}\). If both \(\{g_n\}\) and \(\{h_n\}\) are parameter sequences for \(\{a_n\}\), then

\[
g_n < h_n, \quad n = 1, 2, \ldots, \quad \text{iff} \quad g_0 < h_0.
\]

(14)

Every chain sequence \(\{a_n\}\) has a minimal parameter sequence \(\{m_n\}\) uniquely determined by the condition \(m_0 = 0\), and it has a maximal parameter sequence \(\{M_n\}\) characterized by the fact that \(M_0 > g_0\) for any other parameter sequence \(\{g_n\}\). For every \(x, 0 \leq x \leq m_0\), there is a unique parameter sequence \(\{g_n\}\) for \(\{a_n\}\) such that \(g_0 = x\).

Returning to the context of the birth–death process \(X\), we let

\[
y_n \equiv \frac{β_n}{α_{n-1}α_n}, \quad n = 1, 2, \ldots,
\]

and observe that \(\{y_n\}_{n=1}^\infty\) is a chain sequence, since we can write

\[
y_n = \left(1 - \frac{μ_{n-1}}{λ_{n-1} + μ_{n-1}}\right) \frac{μ_n}{λ_n + μ_n},
\]

so that \(\{μ_n/(λ_n + μ_n)\}\) constitutes a parameter sequence for the chain sequence \(\{γ_n\}\). Our task is now to find all parameter sequences for the chain sequence \(\{γ_n\}\), since there is a one-to-one correspondence between parameter sequences for \(\{γ_n\}\) and sets of birth and death rates satisfying (12). Indeed, for every parameter sequence \(g \equiv \{g_n\}\) we can construct the corresponding birth rates \(\{λ_n^{(g)}\}\) and death rates \(\{μ_n^{(g)}\}\) by letting

\[
λ_n^{(g)} = α_n(1 - g_n), \quad μ_n^{(g)} = α_ng_n, \quad n \in \mathcal{N}.
\]

(15)
The problem of identifying all parameter sequences for a chain sequence for which one parameter sequence is known, has been solved completely by Chihara [5]. (Note that there is a slip in [5, Eq. (3.8)].) In our setting the solution may be formulated as follows.

**Case (i).** \((\mu_0 = 0).\) Let

\[
S_{-1} \equiv 0, \quad S_n \equiv \lambda_0 \sum_{k=0}^{n} (\lambda_k \pi_k)^{-1}, \quad n \in \mathcal{N}, \quad \text{and} \quad S \equiv \lim_{n \to \infty} S_n
\]

(possibly \(S = \infty\)). Then all parameter sequences for \(\{\gamma_n\}\) are given by \(\{g_n(x)\}, \ 0 \leq x \leq 1/S,\) where

\[
g_0(x) = x, \quad g_n(x) = \frac{\mu_n}{\lambda_n + \mu_n} \frac{1 - xS_n - 2}{1 - xS_n - 1}, \quad n \geq 1.
\]

It follows in particular that \(\{\mu_n/(\lambda_n + \mu_n)\}\) is the only parameter sequence for \(\{\gamma_n\}\) if \(S = \infty.\)

**Case (ii).** \((\mu_0 > 0).\) Let

\[
T_{-1} \equiv 0, \quad T_n \equiv \mu_0 \sum_{k=0}^{n} (\mu_k \pi_k)^{-1}, \quad n \in \mathcal{N}, \quad \text{and} \quad T \equiv \lim_{n \to \infty} T_n
\]

(possibly \(T = \infty\)). Then all parameter sequences for \(\{\gamma_n\}\) are given by \(\{g_n(x)\}, \ -\infty \leq x \leq 1/T,\) where

\[
g_n(x) = \frac{\mu_n}{\lambda_n + \mu_n} \frac{1 - xT_n - 1}{1 - xT_n}, \quad n \in \mathcal{N}.
\]

It is interesting to observe that the maximal parameter sequence is obtained for \(x = 1/T.\) So the sequence \(\{\mu_n/(\lambda_n + \mu_n)\}\) is the maximal parameter sequence for \(\{\gamma_n\}\) if \(T = \infty.\)

Translating these results in terms of birth and death rates we obtain the following theorem.

**Theorem 3.** A birth–death process \(X\) with birth rates \(\{\lambda_n, \ n \in \mathcal{N}\}\) and death rates \(\{\mu_n, \ n \in \mathcal{N}\}\) is not similar to any other birth–death process if and only if

\[
\mu_0 = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} = \infty.
\]

In the opposite case the process is similar to any member of an infinite, one-parameter family of birth–death processes \(\{X^{(x)} \mid 0 \leq x \leq 1/S\}\) if \(\mu_0 = 0,\) and \(\{X^{(x)} \mid -\infty \leq x \leq 1/T\}\) if \(\mu_0 > 0.\) The birth rates \(\lambda_n(x), \ n \in \mathcal{N},\) and death rates \(\mu_n(x), \ n \in \mathcal{N},\) of \(X^{(x)}\) are given by

\[
\mu_0(x) \equiv \lambda_0 x, \quad \lambda_n(x) \equiv \lambda_n \frac{1 - xS_n}{1 - xS_{n-1}}, \quad \mu_{n+1}(x) \equiv \mu_{n+1} \frac{1 - xS_n - 1}{1 - xS_n}, \quad n \geq 0,
\]

if \(\mu_0 = 0,\) and

\[
\lambda_n(x) \equiv \lambda_n \frac{1 - xT_{n+1}}{1 - xT_n}, \quad \mu_n(x) \equiv \mu_n \frac{1 - xT_n - 1}{1 - xT_n}, \quad n \geq 0,
\]

if \(\mu_0 > 0.\) The quantities \(S_n, S, T_n\) and \(T\) are defined in (16) and (18).

**Remarks.** The condition (20) for non-similarity as well as the range \(0 = \mu_0(0) \leq \mu_0(x) \leq \mu_0(1/S) = \lambda_0/S\) if \(\mu_0 = 0\) were obtained previously by Karlin and McGregor [8, Lemma 1]. The condition (20) is actually equivalent to the birth–death process \(X\) being recurrent; see, for example, [9]. Within the setting of similarity definition (6) the result for \(\mu_0 > 0\) was obtained earlier by Di Crescenzo [6].
It is interesting to observe that the range of possible values for $\mu_0(x)$, the death rate in state 0 of $\mathcal{X}(x)$, can be represented as

$$0 \leq \mu_0(x) \leq \mu_0 + \left\{ \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \right\}^{-1},$$

both when $\mu_0 = 0$ and when $\mu_0 > 0$. This can easily be verified from the range of possible values for $x$, (21) and (22), and by noting that $\lambda_n \pi_n = \mu_{n+1} \pi_{n+1}$.

3. Invariant vectors and similarity

In this section $\mathcal{X}$ will be a birth–death process with $\mu_0 > 0$. We now let $\mathcal{S} = \mathcal{N} \cup \{-1\}$ be the state space of $\mathcal{X}$ and expand the $q$-matrix $Q$ of $\mathcal{X}$ accordingly, so that $\mathcal{X}$ is conservative on $\mathcal{S}$.

We introduce some terminology and results (see [15] or [17]). A collection of positive numbers $y \equiv \{y_j, j \in \mathcal{S}\}$ is called an invariant vector for $Q$ on $\mathcal{S}$ if

$$\sum_{j \in \mathcal{S}} q_{ij} y_j = 0, \quad i \in \mathcal{S},$$

and it is called an invariant vector for $\mathcal{X}$ on $\mathcal{S}$ if

$$\sum_{j \in \mathcal{S}} p_{ij}(t) y_j = y_i, \quad i \in \mathcal{S}, \quad t \geq 0.$$

It is easy to see that an invariant vector for $\mathcal{X}$ is also an invariant vector for $Q$ on $\mathcal{S}$.

If $y$ is an invariant vector for $\mathcal{X}$ on $\mathcal{S}$, then the functions

$$\tilde{p}_{ij}(t) = \frac{p_{ij}(t) y_j}{y_i}, \quad i, j \in \mathcal{S}, \quad t \geq 0,$$

are the transition functions of an honest process $\tilde{\mathcal{X}}$, the dual of $\mathcal{X}$ with respect to $y$. The elements of the $q$-matrix $\tilde{Q} = (\tilde{q}_{ij}, i, j \in \mathcal{S})$ of $\tilde{\mathcal{X}}$ are given by

$$\tilde{q}_{ij} = \frac{q_{ij} y_j}{y_i}, \quad i, j \in \mathcal{S}.$$

Since $y$ is also invariant for $Q$, it follows that $\tilde{\mathcal{X}}$ is a conservative birth–death process on $\mathcal{S}$.

Clearly, if $y$ is an invariant vector for $\mathcal{X}$ on $\mathcal{S}$, then the dual of $\mathcal{X}$ with respect to $y$ is similar to $\mathcal{X}$. But the reverse is also true, as we show next.

Let $\tilde{\mathcal{X}}$ be honest on $\mathcal{S}$ and similar to $\mathcal{X}$. By Theorem 3 there must be an $x$ in the interval $[-\infty, 1/T]$ such that $\tilde{\mathcal{X}} = \mathcal{X}(x)$ and, correspondingly, $\tilde{\lambda}_n = \lambda_n(x)$ and $\tilde{\mu}_n = \mu_n(x)$. Next defining, for all $j \in \mathcal{S}$, and with $T_j$ as in (18),

$$y_j(x) = \begin{cases} T_j & \text{if } x = -\infty \\ 1 - x T_j & \text{if } -\infty < x \leq \frac{1}{T}, \end{cases}$$

it is readily verified that $y(x) \equiv (y_j(x), j \in \mathcal{S})$ is an invariant vector for $Q$. In fact, it is easy to see that any invariant vector for $Q$ must be of the form (28), apart from normalization. After some algebra it subsequently follows from (28) that

$$y_j(x) = y_0(x) \sqrt{\pi_j / \pi_j}, \quad j \in \mathcal{N},$$
so that, by Theorem 1,
\[ \tilde{p}_{ij}(t) = \frac{p_{ij}(t)y_j(x)}{y_i(x)}, \quad i, j \in \mathcal{N}. \]
(30)

Since \( \tilde{X} \) is honest on \( \mathcal{S} \) we can conclude that \( y(x) \) is actually an invariant vector for \( X \), while \( \tilde{X} \) appears to be the dual of \( X \) with respect to \( y(x) \). Summarizing, we have the following theorem.

**Theorem 4.** Let \( \tilde{X} \) be honest on \( \mathcal{S} \) and similar to \( X \), so that \( \tilde{X} = X(x) \) for some value of \( x \). Then \( \tilde{X} \) is the dual of \( X \) with respect to the invariant vector \( y(x) \equiv (y_j(x), \ j \in \mathcal{S}) \), with \( y_j(x) \) as given in (28).

Interestingly, by Lemma 5 of [17], the absorption probabilities
\[ a_i = \lim_{t \to \infty} \Pr\{X(t) = -1 \mid X(t) = i\}, \quad i \in \mathcal{S}, \]
for the process \( X \) form a positive, invariant vector \( a \equiv (a_i, \ i \in \mathcal{S}) \) for \( Q \). Moreover, if \( X \) is honest, \( y = a \) is the unique minimal, bounded, non-negative solution to (24), normalized so that \( y_{-1} = 1 \). Since any solution to (24) is of the form (28), it is readily seen that the minimal non-negative solution is obtained by choosing \( x = 1/T \) in (28), yielding the well-known result
\[ a_i = 1 - \frac{T_i}{T}, \quad i \in \mathcal{S}; \]
(31)
see [18] and [17] for this and related results.

### 4. Examples

#### 4.1. Example 1. Constant rates

We first want to make some additional remarks on the example already discussed by Di Crescenzo [6]. So let \( X \equiv \{X(t), \ t \geq 0\} \) be the birth–death process with constant birth rates \( \lambda_n \equiv \lambda \) and constant death rates \( \mu_n \equiv \mu, \ n \in \mathcal{N} \). Since \( \mu_0 > 0 \) this process is transient. Ledermann and Reuter [12, formula (4.13)] were the first to show that its transition functions can be represented by
\[ p_{ij}(t) = \left( \frac{\lambda}{\mu} \right)^{j-i} e^{-(\lambda+\mu)t} \left[ I_{j-i}(2t\sqrt{\lambda\mu}) - I_{j+i+2}(2t\sqrt{\lambda\mu}) \right], \ t \geq 0, \]
(32)
where \( I_n(.) \) is the \( n \)th modified Bessel function, see, for example, [1, Sect. 9.6]. It is readily seen that the quantities \( T_n \) of (18) are now given by
\[ T_n = \frac{\lambda^{n+1} - \mu^{n+1}}{(\lambda - \mu)\lambda^n}, \quad n \in \mathcal{N}, \]
(33)
while
\[ T = \begin{cases} \frac{\lambda}{\lambda - \mu} & \text{if } \lambda > \mu \\ \infty & \text{if } \lambda \leq \mu. \end{cases} \]
(34)
Hence, we conclude from Theorem 3 that for each \( x \) in the interval \( -\infty \leq x \leq 1/T \), the process \( X(x) \) with rates
\[ \lambda_n(x) = \lambda + \mu - \mu_n(x), \quad \mu_n(x) = \frac{(\lambda - \mu)\lambda^{n-1} - x(\lambda^n - \mu^n)}{(\lambda - \mu)\lambda^n - x(\lambda^{n+1} - \mu^{n+1})}, \quad n \in \mathcal{N}, \]
(35)
is similar to $\mathcal{X}$, in accordance with [6]. In Figure 1 we show graphs of $\mu_n(x)$ for $n = 0, 1, \ldots, 5$ in the case $\lambda = 2$ and $\mu = 1$.

We will look more closely at the extremal cases $x = 1/T$ and $x = -\infty$. First, if $\lambda \leq \mu$, then $T = \infty$ and hence $\mathcal{X}^{(1/T)} = \mathcal{X}^{(0)} = \mathcal{X}$, the process we started with. But if $\lambda > \mu$, then we have

$$\lambda_n(1/T) = \mu, \quad \mu_n(1/T) = \lambda, \quad n \in \mathcal{N}. \quad (36)$$

So the process $\mathcal{X}^{(1/T)}$ is the process we obtain by interchanging $\lambda$ and $\mu$. Since the time-dependent factors of (32) are symmetric in $\lambda$ and $\mu$, it is evident that $\mathcal{X}^{(1/T)}$ is similar to $\mathcal{X}$.

Taking $x = -\infty$ we readily obtain

$$\lambda_n(-\infty) = \frac{\lambda^{n+2} - \mu^{n+2}}{\lambda^{n+1} - \mu^{n+1}}, \quad \mu_n(-\infty) = \frac{\lambda\mu(\lambda^n - \mu^n)}{\lambda^{n+1} - \mu^{n+1}}, \quad n \in \mathcal{N}, \quad (37)$$

so that $\mu_0(-\infty) = 0$ (as it should be). A little algebra subsequently reveals that the associated transition functions $p_{ij}^{(-\infty)}(t)$ are given by

$$p_{ij}^{(-\infty)}(t) = \lambda^{i-j} \frac{\lambda^{j+1} - \mu^{j+1}}{\lambda^{i+1} - \mu^{i+1}} p_{ij}(t), \quad t \geq 0. \quad (38)$$

4.2. Example 2. Linear rates

We next look at the birth–death process $\mathcal{X} \equiv \{X(t), \ t \geq 0\}$ with birth rates $\lambda_n \equiv n + a$, for some $a > 0$, and death rates $\mu_n \equiv n, \ n \in \mathcal{N}$, so that $\mu_0 = 0$. The transition functions for $\mathcal{X}$ can be represented by

$$p_{ij}(t) = \frac{i!}{\Gamma(a+i)} \int_0^\infty e^{-x(t+1)} L_i^{(a-1)}(x) L_j^{(a-1)}(x) x^{a-1} \, dx, \quad t \geq 0, \quad (39)$$

where $\Gamma(.)$ denotes the gamma function

$$\Gamma(z) \equiv \int_0^\infty e^{-x} x^{z-1} \, dx,$$
and \( L_n^{(a)}(x) \) is the \( n \)th Laguerre polynomial, normalized such that \( L_n^{(a)}(0) = (n+a)! \); see [10] where \( \mathcal{X} \) is said to be of type B. The integrals (39) can easily be evaluated explicitly, yielding

\[
p_{ij}(t) = \frac{i!}{(1+t)^{a+i}} \sum_{k=0}^{i} \frac{(i-1)_k}{k!} \left( \frac{1-t}{t} \right)^k \left( \frac{t}{1+t} \right)^{j-k} \frac{(a+t)_{j-k}}{(j-k)!}, \quad t \geq 0,
\]

for \( i \leq j \), see [10, p. 656], while it is obvious from (39) that

\[
p_{ij}(t) = \frac{i!}{j!} \frac{(a)^j}{(a)^i} p_{ji}(t), \quad t \geq 0.
\]

Here \( (x)_n \equiv \Gamma(x+n)/\Gamma(x) \). It is not difficult to see that condition (20) for non-similarity is satisfied if and only if \( a \leq 1 \), so in what follows we will assume \( a > 1 \).

The quantities \( S_n \) of (16) are now given by

\[
S_n = \sum_{k=0}^{n} \frac{k!}{(a+1)_k} = \frac{1}{a-1} \left( a - \frac{(n+1)!}{(a+1)_n} \right), \quad n \in \mathcal{N},
\]

as can easily be verified by induction. As a consequence,

\[
S = \sum_{k=0}^{\infty} \frac{k!}{(a+1)_k} = \frac{a}{a-1}.
\]

Hence, we conclude from Theorem 3 that for each \( x \) in the interval \( 0 < x < 1 - 1/a \), the process \( \mathcal{X}(x) \) with rates

\[
\mu_0(x) = ax, \quad \mu_n(x) = n(n+a-1) \frac{(a-1)_n - ax[(a)n - (n-1)!]}{(a-1)_{n+1} - ax[(a)n - n!]}, \quad n \geq 1,
\]

and \( \lambda_n(x) = 2n + a - \mu_n(x) \), \( n \in \mathcal{N} \), is similar to \( \mathcal{X} \equiv \mathcal{X}(0) \). In Figure 2 we show graphs of \( \mu_n(x) \) for \( n = 0, 1, 2, \ldots, 10 \) in the case \( a = 2 \).

Looking more closely at the extremal case \( x = 1 - 1/a \), we have

\[
\lambda_n(1 - 1/a) = n + 1, \quad \mu_n(1 - 1/a) = n + a - 1, \quad n \in \mathcal{N}.
\]

Karlin and McGregor [10] have found explicit expressions for the transition functions of the process \( \mathcal{X}(1-1/a) \) (which they call of type A), which are indeed similar to those of \( \mathcal{X} \equiv \mathcal{X}(0) \) and given by

\[
p_{ij}^{(1-1/a)}(t) = \frac{j!}{i!} \frac{(a)^j}{(a)^i} p_{ij}(t).
\]

5. Birth–death processes with finite state space

Let us now consider a birth–death process \( \mathcal{X} \equiv \{X(t), \ t \geq 0\} \) taking values in the finite set \( \mathcal{N} \equiv \{0, 1, \ldots, N\} \) with birth rates \( \{\lambda_n, \ n \in \mathcal{N}\} \) and death rates \( \{\mu_n, \ n \in \mathcal{N}\} \), all strictly positive except \( \mu_0 \) and \( \lambda_N \), which may be equal to 0. When \( \mu_0 > 0 \) the process may escape from \( \mathcal{N} \), via 0, to an absorbing state \(-1\), and when \( \lambda_N > 0 \) the process may escape from \( \mathcal{N} \), via \( N \), to an absorbing state \( N + 1 \).

Again we can pose the question of whether there exist birth–death processes which are similar—in the sense of Definition 1—to \( \mathcal{X} \). This problem may be analysed in a way which is
similar to that of Section 2, but now involving finite chain sequences, that is, numerical sequences \( \{a_n\}_{n=1}^N \) for which there exists a sequence \( \{g_n\}_{n=0}^N \)—a parameter sequence for \( \{a_n\} \)—such that

\[
\begin{align*}
(i) & \quad 0 \leq g_0 < 1, \quad 0 < g_n < 1, \quad n = 1, 2, \ldots, N - 1, \quad 0 < g_N \leq 1, \\
(ii) & \quad a_n = (1 - g_{n-1})g_n, \quad n = 1, 2, \ldots, N.
\end{align*}
\]

We will not give the details of the analysis leading to the following analogue of Theorem 3.

**Theorem 5.** A birth–death process \( \mathcal{X} \) with finite state space \( \mathcal{N} \equiv \{0, 1, \ldots, N\} \), birth rates \( \{\lambda_n, \ n \in \mathcal{N}\} \) and death rates \( \{\mu_n, \ n \in \mathcal{N}\} \) is not similar to any other birth–death process if and only if

\[
\mu_0 = \lambda_N = 0.
\]

In the opposite case the process is similar to an infinite, one-parameter family of birth–death processes \( \{\mathcal{X}^x \mid 0 \leq x \leq (S_{N-1} + \lambda_0 D)^{-1}\} \) if \( \mu_0 = 0 \), and \( \{\mathcal{X}^x \mid -\infty \leq x \leq (T_N + \mu_0 D)^{-1}\} \) if \( \mu_0 > 0 \). Here \( D \equiv (\lambda_N \pi_N)^{-1} \), which should be interpreted as \( \infty \) when \( \lambda_N = 0 \), and the quantities \( S_n \) and \( T_n \) are as in (16) and (18), respectively. The birth rates \( \lambda_n(x), \ n \in \mathcal{N} \) and death rates \( \mu_n(x), \ n \in \mathcal{N} \), of \( \mathcal{X}^x \) are given by (21) if \( \mu_0 = 0 \), and (22) if \( \mu_0 > 0 \), with the exception \( \lambda_N(x) = \lambda_N (1 - x(T_N + \mu_0 D)) \) if \( \mu_0 > 0 \).

It is easy to see that the ranges of possible values of \( \mu_0(x) \) and \( \lambda_N(x) \) are given by

\[
0 \leq \mu_0(x) \leq \mu_0 + \lambda_N \pi_N \left(1 + \lambda_N \pi_N \sum_{n=0}^{N-1} (\lambda_n \pi_n)^{-1}\right)^{-1}
\]

and

\[
0 \leq \lambda_N(x) \leq \lambda_N + \mu_0 \pi_N^{-1} \left(1 + \mu_0 \sum_{n=0}^{N-1} (\lambda_n \pi_n)^{-1}\right)^{-1},
\]

for \( \mu_0 \geq 0 \) and \( \lambda_N \geq 0 \). Moreover, when \( \mu_0(x) \) attains its minimal value 0 then \( \lambda_N(x) \) attains its maximal value and vice versa.
6. v-Similarity

We start this section by stating and proving the generalization of Theorem 1 which was announced in Section 1.

**Theorem 6.** If \( \mathcal{X} \) and \( \tilde{\mathcal{X}} \) are birth–death processes such that \( \tilde{\mathcal{X}} \) is v-similar to \( \mathcal{X} \), then their birth and death rates are related as

\[
\tilde{\lambda}_n + \tilde{\mu}_n = \lambda_n + \mu_n - v, \quad \tilde{\lambda}_n \tilde{\mu}_{n+1} = \lambda_n \mu_{n+1}, \quad n \in \mathcal{N},
\]

while their transition functions satisfy

\[
\tilde{p}_{ij}(t) = \sqrt{\frac{\pi_i \pi_j}{\pi_i \pi_j}} e^{vt} p_{ij}(t), \quad i, j \in \mathcal{N}, \ t \geq 0.
\]

**Proof.** It will be convenient to let \( P(t) \equiv \{p_{ij}(t), \ i, j \in \mathcal{N}\} \), and write the Kolmogorov equations (2) and (3) in matrix notation as

\[
P'(t) = Q(t)P(t) = P(t)Q(t), \quad t \geq 0.
\]

With superscript \((n)\) denoting (elementwise) \(n\)th derivative we then have \( P^{(n)}(0^+) = Q^n \), so if \( \tilde{\mathcal{X}} \) is v-similar to \( \mathcal{X} \) for some real \( v \), that is, if (10) holds true, then

\[
(Q^n)_{ij} = c_{ij} \sum_{k=0}^{N} \binom{n}{k} v^k (Q^{n-k})_{ij}, \quad i, j \in \mathcal{N}, \ n = 0, 1, \ldots
\]

Since \( Q^n \) and \( Q^\alpha \) are \((2n + 1)\)-diagonal matrices it follows that

\[
(Q^{\alpha}_{i,j})_{ij} = c_{ij} (Q^{\alpha}_{i,j})_{ij}, \quad i, j \in \mathcal{N},
\]

from which we readily obtain

\[
c_{ij} = \begin{cases} 
\tilde{\lambda}_i \tilde{\lambda}_{i+1} \ldots \tilde{\lambda}_{j-1} & j > i \\
\lambda_i \lambda_{i+1} \ldots \lambda_{j-1} & j = i \\
\mu_j+1 \mu_{j+2} \ldots \mu_i & j < i.
\end{cases}
\]

Hence, choosing \( n = 1 \) and \( j = i \) in (54) immediately gives us

\[
\tilde{\lambda}_i + \tilde{\mu}_i = \lambda_i + \mu_i - v, \quad i \in \mathcal{N}.
\]

The forward equations for \( \tilde{\mathcal{X}} \) imply

\[
\tilde{p}_{i+2,i+1}(t) = \tilde{\lambda}_i \tilde{p}_{i+2,i}(t) - (\tilde{\lambda}_{i+1} + \tilde{\mu}_{i+1}) \tilde{p}_{i+2,i+1}(t) + \tilde{\mu}_{i+2} \tilde{p}_{i+2,i+2}(t), \quad i \in \mathcal{N},
\]

which, upon substitution of (10), (56) and (57), leads to

\[
p_{i+2,i+1}(t) = \frac{\tilde{\mu}_{i+1}}{\mu_{i+1}} p_{i+2,i}(t) - (\lambda_{i+1} + \mu_{i+1}) p_{i+2,i+1}(t) + \mu_{i+2} p_{i+2,i+2}(t), \quad i \in \mathcal{N}.
\]
But the forward equations for \( \mathcal{X} \) tell us that the coefficient of \( p_{i+2,j}(t) \) should be \( \lambda_i \), so we must have
\[
\tilde{\lambda}_i \tilde{\mu}_{i+1} = \lambda_i \mu_{i+1}, \quad i \in \mathcal{N},
\]
as required. Moreover, as a consequence of (56) we have
\[
c_{ij} = \frac{\pi_i \tilde{\pi}_j}{\pi_i \pi_j} c_{ji},
\]
while (56) and (58) imply \( c_{ij} = c_{ji}^{-1} \) for all \( i, j \in \mathcal{N} \). Combining these results gives us (52), completing the proof.

As in Section 2 we will now assume that the transition functions \( \{p_{ij}(t), i, j \in \mathcal{N}\} \) of the birth–death process \( \mathcal{X} \equiv \{X(t), t \geq 0\} \), with birth rates \( \{\lambda_n, n \in \mathcal{N}\} \) and death rates \( \{\mu_n, n \in \mathcal{N}\} \) are known. Our aim in this section is to identify all birth–death processes that are \( v \)-similar to \( \mathcal{X} \) for some real \( v \). This problem will be solved in three steps. The first step is simply noting that, if \( \mu_0 > 0 \), then, by Theorem 3, there exists a unique process \( \mathcal{X}^* \) with \( \mu_0^* = 0 \) which is similar to \( \mathcal{X} \), and our problem is equivalent to asking for all birth–death processes which are \( v \)-similar to \( \mathcal{X}^* \) for some real \( v \). So it is no restriction to assume \( \mu_0 = 0 \) in what follows. By Theorem 6 we now have to find all real numbers \( v \) and sets of birth rates \( \{\tilde{\lambda}_n, n \in \mathcal{N}\} \) and death rates \( \{\tilde{\mu}_n, n \in \mathcal{N}\} \) such that (51) is satisfied. In the second step, which is described below, we find all such solutions with the property \( \tilde{\mu}_0 = 0 \). The third step is simply the application of Theorem 3 to each of the solutions with \( \tilde{\mu}_0 = 0 \).

For the second step we make use of the polynomials \( Q_n(x) \) recurrently defined by
\[
\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x), \quad n = 1, 2, \ldots,
\]
\[
\lambda_0 Q_1(x) = \lambda_0 - x, \quad Q_0(x) = 1.
\]
(These are the polynomials appearing in Karlin and McGregor’s [8] spectral representation for the transition functions of a birth–death process.) It can be shown, see, for example, [8] and [3], that for all positive \( n \), \( Q_n(x) \) has \( n \) positive, distinct zeros \( x_{n1} < x_{n2} < \cdots < x_{nn} \). These zeros satisfy the separation property
\[
x_{n+1,i} < x_{ni} < x_{n+1,i+1},
\]
so that the limits \( \xi_i \equiv \lim_{n \to \infty} x_{n1} \) exist. Moreover, it is important to observe that
\[
0 \leq \xi_1 < x_{n1}, \quad n = 1, 2, \ldots
\]

**Theorem 7.** For each \( v \leq \xi_1 \) there is a unique solution \( \tilde{\lambda}_n \equiv \lambda_n^{(v)} \) and \( \tilde{\mu}_n \equiv \mu_n^{(v)} \), \( n \in \mathcal{N} \), to the equations (51) satisfying \( \tilde{\mu}_0 = 0 \), which is given by
\[
\lambda_n^{(v)} = \lambda_n \frac{Q_{n+1}(v)}{Q_n(v)}, \quad \mu_n^{(v)} = \mu_n \frac{Q_n(v)}{Q_{n+1}(v)}, \quad n \in \mathcal{N}.
\]

There is no solution to the system (51) if \( v > \xi_1 \).

**Proof:** It can easily be seen that, for any fixed \( v \), the solution of the system of equations (51) satisfying \( \tilde{\mu}_0 = 0 \) can be found recursively and is given by (62). In order for these quantities to define birth and death rates they should all be positive (except \( \tilde{\mu}_0 \)), which, in view of (60) and (61) is the case if and only if \( v \leq \xi_1 \).
Remarks.

(i) The birth–death processes appearing in this lemma form in fact the exponential family corresponding to \( \mathcal{X} \) in the sense of [11], provided \( \mathcal{X} \) is an honest process. The latter condition is not required in our setting.

(ii) The definition of invariant vector of Section 3 may be generalized to that of a \( v \)-invariant vector, cf. [15] and [17]. Results of the type of those of Section 3 may be obtained in this more general setting.

(iii) For representations and bounds for the quantity \( \xi_1 \) see, for example, [16], [19].

Appendix. Proof of Theorem 2

We let

\[
A \equiv \begin{pmatrix}
-(\lambda_0 + \mu_0) & \sqrt{\lambda_0 \mu_1} & 0 & 0 & 0 & \ldots \\
\sqrt{\lambda_0 \mu_1} & -(\lambda_1 + \mu_1) & \sqrt{\lambda_1 \mu_2} & 0 & 0 & \ldots \\
0 & \sqrt{\lambda_1 \mu_2} & -(\lambda_2 + \mu_2) & \sqrt{\lambda_2 \mu_3} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}, \tag{63}
\]

and note that \( A \) can be represented as

\[
A = \Pi^{1/2} Q \Pi^{-1/2}, \tag{64}
\]

where \( \Pi^{1/2} \) and \( \Pi^{-1/2} \) denote the diagonal matrices with entries \( \pi_n^{1/2}, n \in \mathcal{N} \), and \( \pi_n^{-1/2}, n \in \mathcal{N} \), respectively, on the diagonals. It now follows from (53) that

\[
\Pi^{1/2} P'(t) \Pi^{-1/2} = A \Pi^{1/2} P(t) \Pi^{-1/2} = \Pi^{1/2} P(t) \Pi^{-1/2} A,
\]

so that \( R(t) = R_P(t) \) with

\[
R_P(t) \equiv \Pi^{1/2} P(t) \Pi^{-1/2}, \quad t \geq 0, \tag{65}
\]

is a solution to the system

\[
\begin{align*}
R'(t) &= A R(t) = R(t) A, & t \geq 0, \\
R(0) &= I,
\end{align*} \tag{66}
\]

where \( I \) denotes the infinite identity matrix.

When the birth and death rates of \( \tilde{\mathcal{X}} \) are related to those of \( \mathcal{X} \) as in (7) we have

\[
\tilde{\Pi}^{1/2} \tilde{Q} \tilde{\Pi}^{-1/2} = \Pi^{1/2} Q \Pi^{-1/2} = A. \tag{67}
\]

Consequently, arguing as before, we find that \( R_P(t) \equiv \tilde{\Pi}^{1/2} \tilde{P}(t) \tilde{\Pi}^{-1/2} \) is also a solution to (66). If the solution to (66) is unique, we must therefore have

\[
\tilde{\Pi}^{1/2} \tilde{P}(t) \tilde{\Pi}^{-1/2} = \Pi^{1/2} P(t) \Pi^{-1/2},
\]
and hence
\[ \tilde{P}(t) = \Pi^{-1/2} P(t) \Pi^{-1/2} \tilde{\Pi}^{1/2}, \]  
(68)
so that \( \chi \) and \( \tilde{\chi} \) are similar. So it remains to be shown that the solution to (66) is unique.

Obviously, we are only interested in solutions \( R(t) \equiv (r_{ij}(t), i, j \in \mathcal{N}), t \geq 0 \), to (66) which can be represented in terms of the rates and transition functions of some birth–death process, as in (65). Following the approach of [8] it is not difficult to see that such a solution can also be written as
\[ r_{ij}(t) = \int_0^\infty e^{-xt} q_i(x)q_j(x) \psi(dx), \quad i, j \in \mathcal{N}, t \geq 0, \]
where \( \psi \) is a solution of the Stieltjes moment problem (Smp) associated with the matrix \( A \) of (64), and \( q_i(x) \) are the corresponding orthonormal polynomials. As a consequence, there is a unique (relevant) solution to (66) if and only if the Smp associated with \( A \) is determined. From [8, Ch. IV] or [4] we know that this is the case if and only if either \( \mu_0 = 0 \) and (4) holds true, or \( \mu_0 > 0 \) and (9) holds true. This completes the proof of Theorem 2.

As an aside we note the following. Since the matrix \( A \) of (64) is the same for any member of a family of similar birth–death processes, the preceding statement about unicity of the solution to (66) implies that when \( \mu_0 > 0 \), condition (9) must be equivalent to
\[ \sum_{n=0}^{\infty} (\tilde{\lambda}_n + (\tilde{\lambda}_n \tilde{\mu}_n)^{-1}) = \infty, \]  
(69)
where \( \tilde{\lambda}_n \) and \( \tilde{\mu}_n, n \in \mathcal{N} \), constitute the (unique) solution of (12) satisfying \( \tilde{\mu}_0 = 0 \).

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