Applied Probability Trust (26 September 2011)

DOMAIN OF ATTRACTION OF THE QUASI-STATIONARY DISTRIBUTION FOR THE LINEAR BIRTH AND DEATH PROCESS WITH KILLING

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Abstract

The model of linear birth and death processes with killing has been studied by Karlin and Tavaré in [7]. This paper is concerned with three problems in connection with quasi-stationary distribution (QSD) for linear birth-death process $X(t)$ with killing on a semi-infinite lattice of integers. The first is to determine the decay parameter λ_c of $X(t)$. $\lambda_c = \left[(\lambda+\mu+\kappa)^2 - 4\lambda\mu\right]^{\frac{1}{2}}$ where $i\lambda, i\mu, i\kappa$ are called the birth,death and killing rates. The second, more important, problem is to prove the uniqueness of the QSD which is a geometric distribution. It is interesting to find that the unkilled process has a one-parameter family of QSD while the killed process has precisely one QSD. The last is to solve the domain of attraction problem. We obtain that any initial distribution is in the domain of attraction of the unique QSD for $X(t)$. Our study is motivated in part by the population genetics problem.

Keywords: birth and death process; killing; decay parameter; domain of attraction; quasi-stationary distribution; population genetics

2000 Mathematics Subject Classification: Primary 60J27

Secondary 60J80

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1. Introduction

Quasi-stationary distributions (in short, QSDs) for continuous-time Markov chains have recently attracted much attention. As we all know, a complete treatment of the QSD problem for a given family of processes should accomplish two things (see Pakes [10]):
(i) determination of all QSDs; and
(ii) solve the domain of attraction problem, namely, characterize all laws ν such that a given QSD M is a $\nu-$limiting-conditional distribution (in short, LCD).
Although (i) has been addressed for several cases, details about (ii) are known only for finite Markov processes, and for the subcritical Markov Branching Processes. Determination of all QSDs for birth-death processes with killing has been studied by P.Coolen-Schrijner and van Doorn [5]. In particular he shows that either no QSD exists ($\lambda_c = 0$) or there is a continuum of QSDs indexed by x in an interval $(0, \lambda_c]$.

The model of linear birth and death process $X(t)$ with linear killing has been studied by Karlin and Tavaré in [7]. We know that QSDs are rare for $X(t)$; see Pakes [10] for references. In this paper, we prove that the decay parameter $\lambda_c = \left[(\lambda+\mu+\kappa)^2 - 4\lambda\mu\right]^{\frac{1}{2}} > 0$ where $i\lambda, i\mu, i\kappa$ are called the birth,death and killing rates, and there is a unique QSD which is a geometric distribution for $X(t)$. We are interested in the fact that the unkilled process has a one-parameter family of QSD while the killed process has precisely one QSD.

The domain of attraction of the QSD has received much attention. Karlin and Tavaré [7] has showed that if that any initial distribution concentrates all mass at a single state is in the domain of attraction of the unique QSD for $X(t)$. But in this paper we obtain that any initial distribution is in the domain of attraction of the unique QSD for $X(t)$. That is, for any probability measure $M = \{m_i,\ i = 1, 2, \cdots\}$, we have

$$\lim_{t\to\infty} P_M(X(t) = j | T_0 \wedge T_H > t) = \nu_j,\ j \geq 1,$$

where ν_j is a geometric distribution and T_x is the hitting time of x.

Our study of the model is motivated in part by the following problem form population genetics. Consider a population of N individuals, each of which is classified as one of three possible genotypes AA, Aa, aa. A question of some interest, posed originally in

Robertson [11], is: Given that the population currently comprises only the genotypes AA and Aa, how long does it take to produce the first homozygote aa?

The remainder of the paper is organized as follows. After introducing the concepts and collecting some preliminary results in the next section, we obtain our main results.

2. Preliminaries

We focus on the linear birth and death process with linear killing $X(.) = \{X(t), t \geq 0\}$ which has the state space $\{H\} \cup \{0, 1, 2, \cdots\}$, and generator given by

$$q_{ij} = \begin{cases} \lambda_i = i\lambda & \text{if } j = i+1, \\ \mu_i = i\mu & \text{if } j = i-1, \\ \lambda_i + \mu_i + \kappa_i = -i(\lambda + \mu + \kappa) & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\lambda, \mu, \kappa > 0, i > 0$. $\lambda_i, \mu_i, \kappa_i$ are called the birth,death and killing rates. The parameter κ_i may be regarded as the rates of killing into a fictitious state H, say;

$$P\{X(t+h) = H | X(t) = i\} = \kappa_i h + o(h), \ h \downarrow 0.$$

It is not difficult to verify (see, for example, P.Coolen-Schrijner and van Doorn [5]) that there exist unique non-negative numbers $\widetilde{\lambda}_i$ and $\widetilde{\mu}_i, \ i \geq 0$, such that

$$\begin{cases} \lambda_i + \mu_i + \kappa_i = \widetilde{\lambda}_i + \widetilde{\mu}_i, \\ \lambda_i \mu_i = \widetilde{\lambda}_i \widetilde{\mu}_i, \quad i \geq 0. \end{cases} \tag{2}$$

Actually, for the convenience to calculate, we now let $\widetilde{\lambda}_i = ix_1\lambda, \ \widetilde{\mu}_i = ix_2\mu$, respectively, where $0 < x_1 < 1, \ x_2 > 1$. As a consequence,

$$\begin{cases} \lambda + \mu + \kappa = x_1\lambda + x_2\mu, \\ x_1 x_2 = 1. \end{cases} \tag{3}$$

In view of the system of equations (3), we conclude

$$\lambda + \mu + \kappa = x_1\lambda + \frac{\mu}{x_1}. \tag{4}$$

From (4), we find that x_1 is a root of the following equation:

$$\lambda x^2 - (\lambda + \mu + \kappa)\, x + \mu = 0. \tag{5}$$

We let another root of the equation is x_0. It is easy to see from equation (5) that $x_0 = \frac{\mu}{\lambda x_1}$ and $0 < x_1 < 1 < x_0$.

It is clear that either X is absorbed at 0 or killed at H in finite time. Standard probabilistic arguments show that

$$q_i = P\{X(t) \text{ hits } 0 \text{ before } H | X(0) = i\} = x_1^i, \ i \geq 0.$$

So we are led to look at the associated process $\widetilde{X} = \{\widetilde{X}(t), t \geq 0\}$ obtained by conditioning on $\{0\}$ being reached first. The transition probabilities are given by

$$P_{ij}(t) = \widetilde{P}_{ij}(t) x_1^{i-j}, \ i, j \geq 0. \tag{6}$$

Observe that $\widetilde{P}_{i,i+1}(h) = P_{i,i+1}(h) x_1 = i\lambda x_1 h + o(h)$ and $\widetilde{P}_{i,i-1}(h) = \frac{i\mu}{x_1} h + o(h)$. Therefor, $\widetilde{X}(.)$ is a linear birth-death process with transition rates given by

$$\widetilde{\lambda}_i \equiv i\lambda x_1, \ \widetilde{\mu}_i \equiv \frac{i\mu}{x_1} = i\lambda x_0, \ i \geq 0. \tag{7}$$

Clearly, $\widetilde{\lambda}_i = i\lambda x_1 < \widetilde{\mu}_i = i\lambda x_0$.

Write $P_i(\boldsymbol{.}) = P(\boldsymbol{.} \mid X(0) = i)$, and we say that extinction occurs when the process reaches the space H or the state 0. Denote by T_H the hitting time of H and T_0 the hitting time of the state 0. We assume throughout that eventual killing at space H or absorption at state 0 is certain.

3. Main results

This section is concerned with three problems: the first is to determine the decay parameter λ_c of $X(t)$ and the second, more important, problem is to prove the uniqueness of the QSD which is a geometric distribution. The last is to solve the domain of attraction problem.

Determination of decay parameter is critical to the analysis absorbing countable state Markov processes. First at all, we will recall the definition of the decay parameter of an absorbing Markov process. Let $C = \{1, 2, \cdots\}$. For any $i \in C$, let $\lambda_c = -\lim_{t\to\infty} \frac{1}{t} \log P_{ij}(t), \ \ i, j \in C$, which exists, is finite and independent of i, and we call λ_c the decay parameter of C (see Anderson [1]). $\lambda_c > 0$ is necessary for the existence of QSDs.

Theorem 3.1. *For the linear birth and death process with linear killing,*

$$\lambda_c = \left[(\lambda+\mu+\kappa)^2 - 4\lambda\mu\right]^{\frac{1}{2}}.$$

Proof. As we known, for linear birth and death process $\widetilde{X}$ with transition rates given by

$$\widetilde{\lambda}_i \equiv i\lambda x_1,\ \widetilde{\mu}_i \equiv \frac{i\mu}{x_1} = i\lambda x_0,\ i \geq 0.$$

Denote by $\widetilde{\lambda}_c$ the decay parameter of $\widetilde{X}$. Refer to the example 1 of van Doorn [12], $\widetilde{\lambda}_c = \widetilde{\mu} - \widetilde{\lambda}$. Thus we have $\lambda_c = \widetilde{\lambda}_c = \lambda x_0 - \lambda x_1 = \left[(\lambda+\mu+\kappa)^2 - 4\lambda\mu\right]^{\frac{1}{2}}$.

If $\lambda_c > 0$, then there is a one-parameter family of QSDs for linear birth-death process without killing; see van Doorn [12] for references. However, we are interested in the fact that remaining $\lambda_c > 0$, there is a unique QSD for the process with killing. We will prove the important fact in the following theorem.

Let $P_i(\cdot) = P(\cdot \mid X(0) = i)$ and if $M \equiv \{m_i, i \in C\}$ is an initial distribution on C let $P_M = \sum_{i=1}^{\infty} m_i P_i$. A quasi-stationary distribution (in short, QSD) on C is a proper probability distribution $\nu = \{\nu_j, j \geq 1\}$, such that for all $t \geq 0$

$$\nu_j = P_\nu(X(t) = j | T_H \wedge T_0 > t),\ j \geq 1.$$

In other words, a QSD is an initial distribution on C so that the conditional probability of the process being in state j at time t, given that no absorption and no killing have taken place by that time, is independent of t for all j. We call $\nu = \{\nu_j\}$ the limiting condition distribution (in short, LCD) on C if for some initial distribution M on C it satisfies

$$\nu_j = \lim_{t\to\infty} P_M(X(t) = j | T_H \wedge T_0 > t),\ j \geq 1. \tag{8}$$

If we wish to describe the LCD corresponding to a particular initial distribution M, then we usually speak of the M-LCD (if it exists). If (8) holds, we also say that M is attracted to ν ,or is in the domain of attraction of the QSD ν .

We will acquire our results by a geometric distribution with parameter p $(0 < p < 1)$ which is denoted by

$$P(X = k) = p(1-p)^{k-1},\ k \geq 1,$$

and its probability generating function is

$$g(s) = \frac{(1-p)s}{1-sp},\ |s| \leq 1. \tag{9}$$

Recalling to $\widetilde{P}_{ij}(t)$ in the above section, the explicit representation of $\widetilde{P}_{ij}(t)$ given in Anderson [1], Chapter 3, hence, for $|s| \le 1$, its probability generating function

$$\widetilde{G}_i(s,t) = \sum_{j=0}^{\infty} \widetilde{P}_{ij}(t)s^j = \left(\frac{r\lambda x_0 - 1}{r\lambda x_1 - 1}\right)^i, \ i \ge 1, \tag{10}$$

where

$$r = e^{-\lambda(x_0 - x_1)t}\left(\frac{1-s}{\lambda x_0 - \lambda x_1 s}\right).$$

is computed.

So $\widetilde{G}_i(s,t)$ and $G_i(s,t)$ are related to the equation as follows:

$$G_i(s,t) = \sum_{j=0}^{\infty} P_{ij}(t)s^j = \sum_{j=0}^{\infty} x_1^{i-j}\widetilde{P}_{ij}(t)s^j = x_1^i \widetilde{G}_i(sx_1^{-1}, t). \tag{11}$$

Using (10), (11) and the birth and death rates of $\{\widetilde{X}(t), t \ge 0\}$, we have

$$\begin{aligned} G_i(s,t) &= x_1^i \left[\frac{1-\beta+(\beta-\gamma)sx_1^{-1}}{1-\beta\gamma - sx_1^{-1}\gamma(1-\beta)}\right]^i \\ &= \left[\frac{x_0x_1(1-\beta)+s(x_0\beta - x_1)}{x_0 - \beta x_1 - s(1-\beta)}\right]^i, \ i \ge 1, \ |s| \le 1, \end{aligned} \tag{12}$$

where $\beta = e^{-\lambda(x_0-x_1)t}, \gamma = \frac{x_1}{x_0}$.

Lemma 3.1. $\lim_{t\to\infty} \beta^{-1}(G_i(s,t) - x_1^i) = -x_1^{i-1} i A(s), \ i \ge 1,$ *where* $\beta = e^{-\lambda(x_0-x_1)t}$, *and* $A(s) = \frac{(x_0-x_1)(x_1-s)}{x_0-s}, \ 0 \le s < x_0$.

Proof. The proof is identical to that of Karlin and Tavaré [7, Lemma 2].

Remark. We should note the second method in Zhang and Liu [15], then we find the second method is very useful to solve the problem of QSD. We will use this method to approach to the problem in more general settings, for example, the linear birth-death process with immigration and killing.

We summarize Karlin and Tavaré [7], and conclude the following proposition.

Proposition 3.1. *For the linear birth and death process $X(t)$ with linear killing, that any initial distribution concentrates all mass at a single state is in the domain of attraction of a unique QSD which is a geometric distribution ν_j with parameter $\frac{1}{x_0}$, that is,*

$$\lim_{t\to\infty} P_i(X(t) = j | T_0 \wedge T_H > t) = \nu_j, \ i, j \ge 1. \tag{13}$$

From (10) and (12), we have

$$\begin{aligned} G_M(s,t) &= \sum_{j=0}^{\infty} P_M(X(t)=j)s^j = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} m_i P_{ij}(t)s^j = \sum_{i=1}^{\infty} m_i G_i(s,t) \\ &= \sum_{i=1}^{\infty} m_i \left[\frac{x_0x_1(1-\beta)+s(x_0\beta-x_1)}{x_0-\beta x_1 - s(1-\beta)}\right]^i, \quad |s|\le 1. \end{aligned} \tag{14}$$

To produce our result, we give the following important lemma.

Lemma 3.2. $\lim_{t\to\infty}\beta^{-1}(G_M(s,t)-\sum_{i\in C} m_i x_1^i) = -\sum_{i\in C} i m_i x_1^{i-1}A(s),\ i\ge 1,$ *where* x_1 *is the smaller solution of the equation* $\lambda x^2 - (\lambda+\mu+\kappa)x + \mu = 0; 0 < x_1 < 1 < x_0$*. And*

$$A(s) = \frac{(x_0-x_1)(x_1-s)}{x_0-s},\ 0\le s < x_0.$$

Proof. From (14), we have

$$G_M(s,t) = \sum_{i=1}^{\infty} m_i \left[\frac{x_0x_1(1-\beta)+s(x_0\beta-x_1)}{x_0-\beta x_1 - s(1-\beta)}\right]^i = \sum_{i=1}^{\infty}\sum_{k=0}^{i} m_i \binom{i}{k}\left[\frac{x_0x_1(1-\beta)+s(x_0\beta-x_1)}{x_0-\beta x_1 - s(1-\beta)}\right]^k,$$

and so

$$\begin{aligned} G_M(s,t) - \sum_{i\in C} m_i x_1^i &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} m_i\binom{i}{k}\left[\frac{x_0x_1(1-\beta)+s(x_0\beta-x_1)}{x_0-\beta x_1 - s(1-\beta)}\right]^k \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} m_i\binom{i}{k}\left[-\beta A(s)\left\{1-\beta\left(\frac{x_1-s}{x_0-s}\right)\right\}^{-1}\right]^k x_1^{i-k}, \end{aligned}$$

hence

$$\begin{aligned} \lim_{t\to\infty}\beta^{-1}(G_M(s,t)-\sum_{i\in C} m_i x_1^i) &= -\sum_{i=1}^{\infty}\frac{(x_0-x_1)(x_1-s)}{x_0-s} i m_i x_1^{i-1} \\ &= -A(s)\sum_{i\in C} i m_i x_1^{i-1}. \end{aligned}$$

This completes the proof.

We note that $\sum_{i\in C} i x_1^{i-1} = \frac{1}{(1-x_1)^2} < \infty$, then $\sum_{i\in C} i m_i x_1^{i-1} < \infty$. For $0\le s\le 1$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} P_M(X(t)=j|T_0\wedge T_H > t)s^j &= \frac{G_M(s,t)-G_M(0,t)}{G_M(1,t)-G_M(0,t)} \\ &\to \frac{-A(z)\sum_{i\in C} i m_i x_1^{i-1} + A(0)\sum_{i\in C} i m_i x_1^{i-1}}{-A(1)\sum_{i\in C} i m_i x_1^{i-1} + A(0)\sum_{i\in C} i m_i x_1^{i-1}} \\ &= \frac{-A(s)+A(0)}{-A(1)+A(0)} = \frac{s(1-\frac{1}{x_0})}{1-\frac{s}{x_0}} \quad \text{as } t\to\infty, \end{aligned}$$

We can now formulate the following theorem which is one of our main results.

Theorem 3.2. *For the linear birth and death process $X(t)$ with linear killing, there is a unique QSD, which is a geometric distribution ν_j with parameter $\frac{1}{x_0}$, that attracts all initial distributions supports in $C = \{1, 2, \cdots\}$. That is, for any probability measure $M = \{m_i,\ i \in C\}$, we have*

$$\lim_{t\to\infty} P_M(X(t) = j | T_0 \wedge T_H > t) = \nu_j,\ j \geq 1. \tag{15}$$

The theorem shows that for the linear birth and death process $X(t)$ with linear killing, there isn't a one-parameter family of QSDs like the linear birth and death process, but there is precisely one QSD. Any initial distribution is in the domain of attraction of the geometric distribution. Thus we have solved the domain of attraction problem for $X(t)$.

Acknowledgement

The authors acknowledge the support of the Natural Science Foundation of Hunan Province under Grants No.09JJ6006.

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