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BIRTH–DEATH PROCESSES WITH DISASTER AND INSTANTANEOUS RESURRECTION

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Abstract

A new structure with the special property that instantaneous resurrection and mass disaster are imposed on an ordinary birth–death process is considered. Under the condition that the underlying birth–death process is exit or bilateral, we are able to give easily checked existence criteria for such Markov processes. A very simple uniqueness criterion is also established. All honest processes are explicitly constructed. Ergodicity properties for these processes are investigated. Surprisingly, it can be proved that all the honest processes are not only recurrent but also ergodic without imposing any extra conditions. Equilibrium distributions are then established. Symmetry and reversibility of such processes are also investigated. Several examples are provided to illustrate our results.

Keywords: Birth–death process; disaster; instantaneous resurrection; unstable continuous-time Markov chain; existence; uniqueness; recurrence; ergodicity; equilibrium distribution; symmetry; reversibility

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1. Introduction

In this paper, a new model, the birth–death process with disaster and instantaneous resurrection, is investigated. It is a continuous-time Markov chain (CTMC) on the state space $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ whose infinitesimal behavior is governed by the following pre-$q$-matrix.
**Definition 1.1.** A matrix $Q = \{q_{ij}; \ i, j \in \mathbb{Z}_+\}$ is called a birth–death with disaster and instantaneous resurrection pre-$q$-matrix (henceforth referred to as a BDDIR pre-$q$-matrix) if

$$
q_{ij} = \begin{cases} 
  h_j & \text{if } i = 0, \ j \geq 0, \\
  a_1 + d_1 & \text{if } i = 1, \ j = 0, \\
  b_i & \text{if } i \geq 1, \ j = i + 1, \\
  a_i & \text{if } i \geq 2, \ j = i - 1, \\
  d_i & \text{if } i \geq 2, \ j = 0, \\
  -(a_i + b_i + d_i) & \text{if } i = j \geq 1, \\
  0 & \text{otherwise},
\end{cases}
$$

(1.1)

where

$$
a_i > 0, \quad b_i > 0, \quad d_i \geq 0, \quad h_i \geq 0 \quad (i \geq 1)
$$

and

$$
\sum_{j=1}^{\infty} h_j = -h_0 = +\infty.
$$

(1.2)

Although, without loss of generality, we may assume that $d_1 = 0$, we shall not do so; this allows us to give more convenient examples later. By (1.2) we see that the state zero is instantaneous and this immediately leads to a challenging problem which must be answered before any further investigation is carried out, i.e. does there really exist a CTMC whose infinitesimal behaviour is governed by the matrix $Q$ defined above? In connection with this existence problem, we introduce the following definition.

**Definition 1.2.** A BDDIR pre-$q$-matrix $Q$ is called a BDDIR $q$-matrix if there exists a $\mathbb{Z}_+$-valued (homogeneous) Markov process $\{X(t); \ t \geq 0\}$ whose transition function $\{P(t) = \{p_{ij}(t); \ i, j \in \mathbb{Z}_+\}, \ t \geq 0\}$ satisfies

$$
\lim_{t \to 0^+} \frac{P(t) - I}{t} = Q.
$$

(1.3)

If $Q$ is a BDDIR $q$-matrix, then any one of the corresponding Markov processes (which may not be unique) is called a BDDIR process.

It should be pointed out that the study of the above model is of considerable significance in the general theory of CTMCs. Indeed, interest in CTMCs with instantaneous states, called unstable CTMCs, goes back at least as far as Kolmogorov (1951) and Kendall and Reuter (1954) who considered the existence problem for a special pre-$q$-matrix satisfying (1.2). Subsequently, Reuter (1969) considered another example which was slightly more general than the Kolmogorov pre-$q$-matrix. The structures of Kolmogorov’s and Reuter’s examples are quite simple: the pre-$q$-matrices restricted to $\mathbb{Z}_{++} = \mathbb{Z}_+ \setminus \{0\}$ are just diagonal matrices whose properties are easily treated. These Kolmogorov–Reuter pre-$q$-matrices can be viewed as the cases when the birth–death structure is removed from (1.1). Adding such birth–death structure substantially increases the interest and importance of the model.

Interest in unstable CTMCs can also be found in Chung (1967), Freedman (1983), Rogers and Williams (1986), (1987), (1994), Williams (1967), (1976), Kingman (1972), Chen and Renshaw (1990), (1993), (1995), (2000) and Chen and Liu (2003). However, in most of the literature cited above, the main interest concentrates on the existence problem. In contrast
with those mentioned above, the existence problem is not the only interest to us. In fact, our main interest focuses on probabilistic properties such as recurrence, ergodicity, equilibrium distributions and reversibility for such an abundant and challenging structure. Note also that our model is a generalization of that of Chen and Liu (2003). Indeed, letting \( d_i = 0 \ (i \geq 1) \) will recover their model. One of the main advantages of our current model, compared with that of Chen and Liu (2003), is that it enables us to study the important topic of reversibility.

It is worth pointing out that to study the proposed model is not only of importance in theory but also has significance in applications. The interpretation of the model (1.1), together with (1.2), is that, when the Markov process (if any) approaches a special state, zero, say, it could jump to any other state with any intensity having no limitation. The process may also instantaneously jump back to this special state from any other state. In practice, there exists a variety of natural phenomena which may be modelled in this manner.

The structure of this paper is as follows. The existence criteria are given in Section 3 while the uniqueness criteria and construction of the processes are presented in Section 4. The properties of the corresponding processes, particularly the properties involving recurrence, ergodicity and equilibrium distributions, are discussed in Section 5. The important reversibility is presented in Section 6. Some examples are provided in Section 7 to illustrate our results. It will be seen that all the above discussions are closely linked with some underlying nonconservative birth–death processes and, thus, as a preliminary, these processes are discussed in Section 2. However, some detailed proofs of the results in Section 2 are postponed to Appendix A. Because of the complexity of the model, arguments in this paper will mainly focus on the case of \( R < \infty \) (defined below). The case of \( R = +\infty \) will be discussed in a subsequent paper. However, in Section 2 (and thus also in Appendix A) this assumption will not be imposed. The terminology, notation and the basic results used in this paper will mainly follow Anderson (1991), M. F. Chen (1992) and Yang (1990).

### 2. Nonconservative birth–death processes

For any BDDIR pre-\( q \)-matrix \( Q \) as in (1.1) and (1.2), we shall always denote by \( Q^* \) the restriction of \( Q \) to \( \mathbb{Z}_{++} = \mathbb{Z}_+ \setminus \{0\} \). Hence, \( Q^* \) is an ordinary (nonconservative) birth–death \( q \)-matrix. Following Feller (1959), we define the natural scale \( \{\rho_n; \ n \in \mathbb{Z}_{++}\} \) and the potential coefficients \( \Pi = \{\pi_n; \ n \in \mathbb{Z}_{++}\} \) with respect to \( Q^* \) as follows:

\[
\rho_1 = \frac{1}{a_1},
\rho_2 = \rho_1 + \frac{1}{b_1},
\rho_n = \rho_{n-1} + \frac{a_2a_3 \cdots a_{n-1}}{b_1b_2 \cdots b_{n-1}} \quad (n \geq 3)
\]

and

\[
\pi_1 = 1,
\pi_n = \frac{b_1b_2 \cdots b_{n-1}}{a_2a_3 \cdots a_n} \quad (n \geq 2).
\]

Note that, by (2.1), \( \rho_n \uparrow \infty \) as \( n \to \infty \), and thus the limit \( \rho_\infty = \lim_{n \to \infty} \rho_n \) exists and

\[0 < \rho_\infty \leq \infty.\]
Here and elsewhere, we use ↑↑ to denote strictly increasing. Similarly, ↓↓ means strictly decreasing. Furthermore, define

\[
\mu_1 = \frac{1}{b_1} = (\rho_2 - \rho_1) \pi_1, \quad (2.3)
\]

\[
\mu_j = \frac{1}{b_j} + \sum_{k=1}^{j-1} \frac{a_j a_{j-1} \cdots a_{j-k+1}}{b_j b_{j-1} \cdots b_{j-k}} = (\rho_{j+1} - \rho_j) \sum_{k=1}^{j} \pi_k \quad (j \geq 2), \quad (2.4)
\]

\[
g_j = \sum_{k=j}^{\infty} \mu_k \quad (j \geq 1). \quad (2.5)
\]

In particular, define

\[
R_e \equiv g_1 = \sum_{n=1}^{\infty} \left( \frac{1}{b_n} + \frac{a_n}{b_n b_{n-1}} + \frac{a_n a_{n-1}}{b_n b_{n-1} b_{n-2}} + \cdots + \frac{a_n a_{n-1} \cdots a_2}{b_n b_{n-1} \cdots b_1} \right). \quad (2.6)
\]

It is trivial to see that either every \( g_j, \ j \geq 1, \) is finite or none is. Hence, \( R_e \) is finite if and only if every \( g_j, \ j \geq 1, \) is finite.

Also, define

\[
\pi_{\sigma} = \sum_{k=1}^{\infty} \pi_k. \quad (2.7)
\]

In parallel with \( R_e \) defined in (2.6), we also define \( R_d \) by

\[
R_d = \sum_{n=1}^{\infty} \left( \frac{d_n}{b_n} + \frac{a_n d_{n-1}}{b_n b_{n-1}} + \frac{a_n a_{n-1} d_{n-2}}{b_n b_{n-1} b_{n-2}} + \cdots + \frac{a_n a_{n-1} \cdots a_2 d_1}{b_n b_{n-1} \cdots b_1} \right). \quad (2.8)
\]

Then, let \( R = R_e + R_d, \) i.e.

\[
R = \sum_{n=1}^{\infty} \left( \frac{1 + d_n}{b_n} + \frac{a_n (1 + d_{n-1})}{b_n b_{n-1}} + \frac{a_n a_{n-1} (1 + d_{n-2})}{b_n b_{n-1} b_{n-2}} + \cdots + \frac{a_n a_{n-1} \cdots a_2 (1 + d_1)}{b_n b_{n-1} \cdots b_1} \right). \quad (2.9)
\]

The following five lemmas, especially the construction Lemma 2.4, will play a crucial role in the analysis of BDDIR processes. However, we shall only state the lemmas here. The detailed proofs will be postponed to Appendix A, except for the simple Lemma 2.5.

**Lemma 2.1.** For each \( \lambda > 0, \) the equation

\[
(\lambda I - Q^*) U(\lambda) = 0 \quad (2.10)
\]

has exactly one solution (up to constant multiples). That is, each solution can be expressed as \( \{c(\lambda) u_n(\lambda), \ n \geq 1\}, \) where \( u_n(\lambda) \) is recursively given by

\[
u_1(\lambda) = 1,
\]

\[
u_n(\lambda) = a_1 \rho_n + \sum_{j=1}^{n-1} (\rho_n - \rho_j)(\lambda + d_j) u_j(\lambda) \pi_j \quad (n \geq 2) \quad (2.11)
\]

and \( c(\lambda) \) is a constant. Moreover, the solution \( U(\lambda) = \{u_n(\lambda); \ n \in \mathbb{Z}_+\} \) given in (2.11) has the following properties:
(i) For each $\lambda > 0$, $u_n(\lambda)$ is a strictly increasing function of $n$ and thus the limit

$$u_n(\lambda) \uparrow u_\infty(\lambda) \quad (n \to \infty)$$

exists and $u_\infty(\lambda) < \infty$ (for all $\lambda > 0$) if and only if $R < +\infty$, where $R$ is given in (2.9).

(ii) For each $n \in \mathbb{Z}_{++}$, $u_n(\lambda)$ is a strictly increasing function of $\lambda > 0$, and thus the limit

$$\lim_{\lambda \to 0} u_n(\lambda) = u_n(0)$$

exists. These limits $u_n(0)$ can be recursively given by

$$u_1(0) = 1,$$

$$u_n(0) = a_1 \rho_n + \sum_{j=1}^{n-1} (\rho_n - \rho_j) \delta_d u_j(0) \pi_j \quad (n \geq 2).$$

(iii) If $R < \infty$, the finite-valued function $u_\infty(\lambda)$ is also a strictly increasing function of $\lambda > 0$ such that

$$u_\infty(\lambda) \downarrow a_1 \rho_\infty + \sum_{j=1}^{\infty} (\rho_\infty - \rho_j) \delta_d u_j(0) \pi_j \quad (\lambda \to 0),$$

where $\rho_\infty < \infty$ is guaranteed by the condition $R < \infty$.

(iv) If $R < \infty$, then the dimension of the solution space of the equation

$$(\lambda I - Q^*) U(\lambda) = 0, \quad 0 \leq U(\lambda) \leq 1,$$

is exactly 1, and the maximum solution of (2.14), denoted by $\tilde{U}(\lambda) = \{\tilde{u}_n(\lambda); \ n \geq 1\}$, is given by

$$\tilde{u}_n(\lambda) = \frac{u_n(\lambda)}{u_\infty(\lambda)}.$$  

(Here and elsewhere, 0 and 1 denote the column vectors whose components are all 0 and all 1 respectively.) If $R = \infty$, then (2.14) has only the trivial solution $U(\lambda) \equiv 0$.

**Lemma 2.2.** For each fixed $\lambda > 0$, the equation

$$W(\lambda)(\lambda I - Q^*) = 0$$

has exactly one solution (up to constant multiples). That is, each solution can be expressed as $c(\lambda) w_n(\lambda)$, where $w_n(\lambda)$ is given by

$$w_n(\lambda) = u_n(\lambda) \pi_n \quad (n \in \mathbb{Z}_{++})$$

and $u_n(\lambda)$ and $\pi_n$ are given in (2.11) and (2.2) respectively. Moreover, we have the following conclusions:

(i) For each $n \geq 1$, $w_n(\lambda)$ is a strictly increasing function of $\lambda > 0$ such that

$$w_n(\lambda) \downarrow u_n(0) \pi_n \quad (\lambda \to 0).$$

(ii) The dimension of the solution space of the equation

$$\eta(\lambda)(\lambda I - Q^*) = 0, \quad 0 \leq \eta(\lambda) \in l_1,$$

where $l_1 = \{(f_n) : \sum_{n=1}^{\infty} |f_n| < \infty\}$, is independent of $\lambda > 0$, and thus may be denoted by $\mathcal{N}^+(Q^*)$. Moreover, $\mathcal{N}^+(Q^*)$ must be either 1 or 0.
(iii) If \( R < \infty \), then whether \( \mathcal{N}^+(Q^*) \) is 1 or 0 depends upon \( \pi_\sigma \) being finite or not, where \( \pi_\sigma \) is given in (2.7). In more detail, if \( \pi_\sigma = \infty \), then (2.18) has only the trivial solution \( \eta(\lambda) \equiv 0 \) while, if \( \pi_\sigma < \infty \), then \( \mathcal{N}^+(Q^*) = 1 \).

(iv) If \( R < \infty \) and \( \pi_\sigma < \infty \), then \( \sum_{n=1}^{\infty} w_n(\lambda) \) is also a finite increasing function of \( \lambda > 0 \) and thus

\[
\sum_{n=1}^{\infty} w_n(\lambda) \downarrow \sum_{n=1}^{\infty} u_n(0)\pi_n \quad (\lambda \to 0).
\]  

Now define

\[
v_n(\lambda) = u_n(\lambda) \sum_{j=n}^{\infty} \frac{\rho_{j+1} - \rho_j}{u_j(\lambda)u_{j+1}(\lambda)} \quad (n \geq 1).
\]

**Lemma 2.3.** (i) The function \( v_n(\lambda) \) given in (2.20) is well defined for all \( n \geq 1 \) and \( \lambda > 0 \), i.e. the series defined on the right-hand side of (2.20) is convergent for all \( n \geq 1 \) and \( \lambda > 0 \).

(ii) For any \( \lambda > 0 \), \( v_n(\lambda) \) is a strictly decreasing function of \( n \) and the following relations hold:

\[
\frac{v_{n+1}(\lambda) - v_n(\lambda)}{\rho_{n+1} - \rho_n} \uparrow \uparrow - \frac{1}{u_\infty(\lambda)}, \quad n \to \infty,
\]

and

\[
(\lambda + a_1 + b_1 + d_1)v_1(\lambda) - b_1v_2(\lambda) = 1
\]

\[
\lambda v_n(\lambda) - a_n v_{n-1}(\lambda) + (a_n + b_n + d_n) v_n(\lambda) - b_n v_{n+1}(\lambda) = 0, \quad n \geq 2.
\]

**Lemma 2.4.** For the above q-matrix \( Q^* \), the Feller minimal \( Q^* \)-resolvent \( \Phi^*(\lambda) = \{\phi^*_ij(\lambda); i, j \in \mathbb{Z}_+\} \) is given by

\[
\phi^*_ij(\lambda) = \begin{cases} 
 u_i(\lambda)v_j(\lambda)\pi_j & \text{if } i \leq j, \\
 v_i(\lambda)u_j(\lambda)\pi_j & \text{if } i \geq j,
\end{cases}
\]

where \( \{u_n(\lambda); n \geq 1\} \), \( \{v_n(\lambda); n \geq 1\} \) and \( \{\pi_n; n \geq 1\} \) are given in (2.11), (2.20) and (2.2) respectively. Moreover, \( \Phi^*(\lambda) \) constructed in (2.24) possesses the following properties:

(i) We have

\[
\lambda \sum_{j=1}^{\infty} \phi^*_ij(\lambda) = 1 - a_1v_1(\lambda) - \sum_{j=1}^{\infty} \phi^*_ij(\lambda)d_j - u_1(\lambda)\frac{1}{u_\infty(\lambda)},
\]

where, as is the usual convention, the term \( 1/u_\infty(\lambda) \) on the right-hand side is taken to be zero if \( u_\infty(\lambda) = \infty \) for \( \lambda > 0 \) (equivalently, \( R = \infty \)).

(ii) If \( R < \infty \) (and thus \( \rho_\infty < \infty \); see Lemma 2.5), then, when \( \lambda \to 0 \),

\[
\phi^*_ij(\lambda) \uparrow \begin{cases} 
 \pi_ju_i(0)u_j(0) \sum_{n=j}^{\infty} \frac{\rho_{n+1} - \rho_n}{u_n(0)u_{n+1}(0)} & \text{if } j \geq i, \\
 \pi_ju_i(0)u_j(0) \sum_{n=i}^{\infty} \frac{\rho_{n+1} - \rho_n}{u_n(0)u_{n+1}(0)} & \text{if } j \leq i,
\end{cases}
\]

\[
\lambda \sum_{j=1}^{\infty} \phi^*_ij(\lambda) \downarrow 0 \quad (i \geq 1)
\]
and
\[ \sum_{j=1}^{\infty} \phi_{ij}^*(\lambda) \uparrow m_i < \infty \quad (i \geq 1), \] (2.28)

where
\[ m_i = u_i(0) \sum_{n=1}^{\infty} \frac{\rho_n + 1 - \rho_n}{u_n(0)u_{n+1}(0)} \sum_{j=1}^{n} \pi_j u_j(0) \quad (i \geq 1). \] (2.29)

**Remark 2.1.** By (2.24) it is easy to see that the Feller minimal \( Q^* \)-resolvent constructed in Lemma 2.4 satisfies the following relations:
\[ \pi_i \phi_{ij}^*(\lambda) = \pi_j \phi_{ji}^*(\lambda) \quad (i, j \geq 1, \lambda > 0). \] (2.30)

Such relations, usually called the weak symmetric property, are well known for ordinary birth-death processes and will also play an essential role in analysing properties of our BDDIR processes. See Section 6 below.

**Remark 2.2.** In Lemma 2.2(iii), it is stated that, if \( R < \infty \) and \( \pi_\sigma < \infty \), then the dimension of the solution space of (2.18) is exactly 1. Given Lemma 2.4, it can be easily verified that each solution \( \tilde{\eta}(\lambda) = \{\tilde{\eta}_j(\lambda); j \geq 1\} \) of (2.18) which satisfies the further condition that
\[ \tilde{\eta}(\lambda) - \tilde{\eta}(\mu) = (\mu - \lambda)\tilde{\eta}(\lambda)\Phi^*(\mu) \quad (\lambda, \mu > 0) \] (2.31)
can be expressed as
\[ \tilde{\eta}_j(\lambda) = c\bar{w}_j(\lambda) \equiv cw_j(\lambda)\frac{1}{u_\infty(\lambda)}, \quad j \geq 1, \] (2.32)
where \( c \geq 0 \) is a constant, \( \{w_j(\lambda); \ j \geq 1\} \) and \( u_\infty(\lambda) \) are given in (2.16) and Lemma 2.1(i) respectively and \( \Phi^*(\lambda) \) is the Feller minimal \( Q^* \)-resolvent constructed in Lemma 2.4. Naturally, \( \{\bar{w}_j(\lambda); \ j \geq 1\} \) itself satisfies (2.31).

**Lemma 2.5.** Suppose that \( R_\varepsilon < \infty \), where \( R_\varepsilon \) is given in (2.6). Then \( \rho_\infty < \infty \) and the terms \( g_j, \ j \geq 1 \), defined in (2.5) can be rewritten as follows:
\[ g_j = (\rho_\infty - \rho_j) \sum_{k=1}^{j} \pi_k + \sum_{k=j+1}^{\infty} (\rho_\infty - \rho_k)\pi_k. \] (2.33)

**Proof.** Comparing (2.6) with (2.1) we see that \( R_\varepsilon < \infty \) implies that \( \rho_\infty < \infty \), which, together with (2.4), in turn implies that \( g_j \) in (2.5) can be rewritten as (2.33).

### 3. Existence of BDDIR processes

The first basic and important task is to establish an existence criterion. More precisely, given a BDDIR pre-q-matrix \( Q \), we need to establish conditions under which there exists a \( Q \)-function which satisfies (1.3) and, then, by the general theory of CTMCs, a corresponding Markov process \( \{X(t); \ t \geq 0\} \) can be constructed. As usual, we shall call this Markov process (if any) a \( Q \)-process. Note that, in accordance with Reuter (1959), (1962), a \( Q \)-process may refer to the \( Q \)-function \( P(t) \), the \( Q \)-resolvent \( R(\lambda) \) or the Markov process \( \{X(t); \ t \geq 0\} \) itself. In cases where we need to distinguish them, they can be clearly recognized by the above notation.
Since a BDDIR pre-$q$-matrix $Q$ is a special case of the so-called uni-conservative pre-$q$-matrix discussed by Chen and Renshaw (1993), their results immediately yield the following existence criterion. Here and elsewhere, $H = \{h_j; j \geq 1:\}$

**Theorem 3.1.** (Existence criterion.) Suppose that $Q$ is a BDDIR pre-$q$-matrix. The following statements are equivalent:

(i) There exists a $Q$-process.

(ii) There exists an honest $Q$-process.

(iii) For any (equivalently, for some) $\lambda > 0$, the relation

$$H \Phi^*(\lambda) 1 < +\infty,$$

that is,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} h_i \phi_{ij}^*(\lambda) < +\infty,$$

holds, where $\Phi^*(\lambda) = \{\phi_{ij}^*(\lambda); i, j \geq 1\}$ is the Feller minimal $Q^*$-resolvent constructed in Lemma 2.4.

The condition (3.2) is not very satisfactory since it involves the $Q^*$-resolvent rather than the elements of the given $Q$. From Lemma 2.4, we see that the construction of the $Q^*$-resolvent is quite complicated and thus verifying (3.2) directly is not an easy task. A simple criterion is therefore needed, particularly from the viewpoint of applications. Fortunately, such an aim can be achieved under the assumption that $R < \infty$. For this and other reasons, from now on we shall always assume that $R < \infty$. The more subtle case, $R = \infty$, shall be discussed in a subsequent paper.

In connection with this assumption, we give the following definition.

**Definition 3.1.** Let $Q$ be a given BDDIR pre-$q$-matrix.

(i) We say that $Q$ is bilateral if $R < \infty$ and $\pi_\sigma < \infty$.

(ii) We say that $Q$ is exit if $R < \infty$ and $\pi_\sigma = \infty$.

Note that the above classification is just the Feller boundary classification with respect to $Q^*$. However, for simplicity, we have applied the terms to $Q$ itself. In order to avoid possible confusion, we use the new term ‘bilateral’ rather than ‘regular’, as was originally used by Feller.

Now, under the basic assumption $R < \infty$, we obtain the following equivalent criterion which is easy to check.

**Theorem 3.2.** For a BDDIR pre-$q$-matrix $Q$ satisfying $R < +\infty$, there exists a $Q$-process if and only if

$$\sum_{j=1}^{\infty} h_j g_j < \infty,$$

where $h_j$ and $g_j$ for $j = 1, 2, \ldots$ are given in (1.1) and (2.33) respectively.
Proof. By Lemma 2.1 we know that $u_{\infty}(\lambda) < \infty$ (for $\lambda > 0$) if and only if $R < \infty$ and thus $R < \infty$ implies that $1/u_{\infty}(\lambda)^2 > 0$. Now, comparing (3.2) with (3.3) shows that, if we can prove that

$$\frac{1}{u_{\infty}(\lambda)^2} g_i \leq \sum_{j=1}^{\infty} \phi^*_{ij}(\lambda) \leq g_i \quad (i \geq 1),$$

then Theorem 3.2 immediately follows. In order to prove (3.4), recall that $u_n(\lambda)$ in (2.11) is a strictly increasing function of $n$ for each $\lambda > 0$ and thus $v_n(\lambda)$ defined in (2.20) satisfies the inequality

$$\frac{1}{u_{\infty}(\lambda)^2} u_i(\lambda)(\rho_{\infty} - \rho_i) \leq v_i(\lambda) \leq \frac{1}{u_i(\lambda)} (\rho_{\infty} - \rho_i) \quad (i \geq 1).$$

This inequality, together with (2.24) and the fact that $u_n(\lambda)$ is increasing with respect to $n$, immediately yields that, for all $i \geq 1$,

$$\sum_{j=1}^{\infty} \phi^*_{ij}(\lambda) \leq \frac{1}{u_i(\lambda)} (\rho_{\infty} - \rho_i) \sum_{j=1}^{i} u_i(\lambda) \pi_j + u_i(\lambda) \sum_{j=i+1}^{\infty} \frac{1}{u_j(\lambda)} (\rho_{\infty} - \rho_j) \pi_j$$

$$\leq (\rho_{\infty} - \rho_i) \sum_{j=1}^{i} \pi_j + u_i(\lambda) \frac{1}{u_i(\lambda)} \sum_{j=i+1}^{\infty} (\rho_{\infty} - \rho_j) \pi_j = g_i$$

and

$$\sum_{j=1}^{\infty} \phi^*_{ij}(\lambda) \geq \frac{1}{u_{\infty}(\lambda)^2} u_i(\lambda)(\rho_{\infty} - \rho_i) \sum_{j=1}^{i} u_i(\lambda) \pi_j + \frac{1}{u_{\infty}(\lambda)^2} u_i(\lambda) \sum_{j=i+1}^{\infty} u_i(\lambda)(\rho_{\infty} - \rho_j) \pi_j$$

$$= \frac{1}{u_{\infty}(\lambda)^2} u_i(\lambda) g_i \geq \frac{1}{u_{\infty}(\lambda)} u_1(\lambda) g_i.$$

Since $u_1(\lambda) = 1$, the desired inequality (3.4) is now proved.

If $Q$ is bilateral, we may get a much simpler criterion, as the following theorem shows.

**Theorem 3.3.** Suppose that $Q$ is a BDDIR pre-$q$-matrix satisfying $R < \infty$. If there exists a $Q$-process, then

$$\sum_{j=1}^{\infty} h_j(\rho_{\infty} - \rho_j) < \infty. \quad (3.6)$$

Moreover, if $Q$ is bilateral, then there exists a $Q$-process if and only if (3.6) holds.

Proof. By (2.4) and (2.5), it is easy to see that

$$g_j = \sum_{k=j}^{\infty} \mu_k \geq \sum_{k=j}^{\infty} (\rho_{k+1} - \rho_k) \pi_1 = \rho_{\infty} - \rho_j,$$

and thus (3.3) implies (3.6). On the other hand, if $Q$ is bilateral, then $\pi_\sigma < \infty$, and thus (2.4) and (2.5) give

$$\rho_{\infty} - \rho_j \leq g_j \leq \sum_{k=j}^{\infty} (\rho_{k+1} - \rho_k) \pi_\sigma = \pi_\sigma(\rho_{\infty} - \rho_j),$$

and so (3.3) and (3.6) are equivalent.
4. Uniqueness and construction

Now we come to the second fundamental problem, the uniqueness of the $Q$-processes.

**Theorem 4.1.** Suppose that $R < \infty$ and $Q$ is a BDDIR $q$-matrix, i.e. (3.3) holds. Then

(i) there always exist infinitely many $Q$-processes;

(ii) there exists only one honest $Q$-process if and only if $\pi_\sigma = \infty$, i.e. $Q$ is exit. Moreover, if $Q$ is bilateral, then there exist infinitely many honest $Q$-processes.

**Proof.** The claim (i) is well known (see Section 5 of Chen and Renshaw (1993)). In order to prove (ii), we first prove that the equation

$$(\lambda I - Q^*)X(\lambda) = 0,$$

as $(4.1)$ holds. By Lemma 2.1, the equation

$$(\lambda I - Q^*)U(\lambda) = 0, \quad 0 \leq U(\lambda) \leq 1,$$

has only the trivial solution for any $\lambda > 0$. By Lemma 2.1, the equation

$$(\lambda I - Q^*)X(\lambda) = 0, \quad 0 < X(\lambda) < 1,$$

has only the independent solution $X(\lambda) = \{x_i(\lambda); i \in \mathbb{Z}_+\}$ which satisfies the condition $x_i(\lambda) \uparrow 1$ as $i \to \infty$ (see (2.15)). This solution of course cannot satisfy $HX(\lambda) < \infty$ since $H1 = \infty$. Thus, (4.1) has only the trivial solution. Now, by Theorem 5.1 of Chen and Renshaw (1993), we see that in order to complete the proof we only need to prove that the equation

$$Y(\lambda)(\lambda I - Q^*) = 0, \quad 0 \leq Y(\lambda) \in l_1,$$

has only the trivial solution if and only if $\pi_\sigma = \infty$. However, this fact immediately follows from Lemma 2.2 and the assumption that $R < \infty$ (see Lemma 2.2(iii)). The last statement in (ii) is well known; see Hou and Guo (1988) or Yang (1990).

We now consider the construction of all honest $Q$-resolvents with respect to a BDDIR $q$-matrix $Q$. First, assume that $\pi_\sigma = \infty$. Then, by Theorem 4.1, there exists only one honest $Q$-process.

**Theorem 4.2.** Suppose that the BDDIR $q$-matrix $Q$ is exit. Then the (unique) honest $Q$-resolvent $R(\lambda) = \{r_{ij}(\lambda); i, j \in \mathbb{Z}_+\}$ can be represented as follows:

$$r_{00}(\lambda) = \left(\lambda + \lambda \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} h_{l, k}(\lambda)\right)^{-1},$$

$$(4.4)$$

$$r_{0j}(\lambda) = r_{00}(\lambda) \sum_{k=1}^{\infty} h_{k, j}(\lambda) \quad (j \geq 1),$$

$$(4.5)$$

$$r_{i0}(\lambda) = \left(1 - \lambda \sum_{k=1}^{\infty} \phi_{ik}(\lambda)\right)r_{00}(\lambda) \quad (i \geq 1),$$

$$(4.6)$$

$$r_{ij}(\lambda) = \phi_{ij}(\lambda) + \left(1 - \lambda \sum_{k=1}^{\infty} \phi_{ik}(\lambda)\right)r_{00}(\lambda) \sum_{k=1}^{\infty} h_{k, j}(\lambda) \quad (i, j \geq 1),$$

$$(4.7)$$

where $\Phi^*(\lambda) = \{\phi_{ij}(\lambda); i, j \geq 1\}$ is the Feller minimal $Q^*$-resolvent given in Lemma 2.4.
Proof. This is a direct consequence of Theorem 6.1 of Chen and Renshaw (1993).

It is interesting to note that, if \( Q \) is exit, then the structure of the honest \( Q \)-resolvent \( R(\lambda) \) is totally determined by the \( Q^* \)-resolvent \( \Phi^*(\lambda) \) and the given \( \{h_j; \ j \geq 1\} \), and thus is very simple. More interestingly, if the given BDDIR \( q \)-matrix \( Q \) is bilateral, then, although there exist infinitely many of them, all these honest \( Q \)-processes have a very simple structure as the following theorem shows. This unexpected result is very useful in analysing properties of the corresponding \( Q \)-processes.

**Theorem 4.3.** Suppose that the BDDIR \( q \)-matrix \( Q \) is bilateral. Firstly, choose any constant \( c \geq 0 \) and let

\[
\begin{align*}
\eta(\lambda) &= H \Phi^*(\lambda) + c \tilde{W}(\lambda) \quad (\lambda > 0), \\
\zeta(\lambda) &= 1 - \lambda \Phi^*(\lambda) \ 1, \quad (4.8)
\end{align*}
\]

where \( \eta(\lambda) = \{\eta_j(\lambda); \ j \geq 1\} \) and \( \zeta(\lambda) = \{\zeta_j(\lambda); \ j \geq 1\} \). Here \( H = \{h_j; \ j \geq 1\} \), \( \tilde{W}(\lambda) = \{\tilde{w}_n(\lambda); \ n \geq 1\} \) and the Feller minimal \( Q^* \)-resolvent \( \Phi^*(\lambda) \) are given by \((1.1), (2.32)\) and \((2.24)\) respectively. Secondly, let

\[
\begin{align*}
r_{00}(\lambda) &= (\lambda + \lambda \eta(\lambda) \ 1)^{-1}, \\
r_{0j}(\lambda) &= r_{00}(\lambda) \eta_j(\lambda) \quad (j \geq 1), \\
r_{i0}(\lambda) &= \zeta_i(\lambda) r_{00}(\lambda) \quad (i \geq 1), \\
r_{ij}(\lambda) &= \phi^*_{ij}(\lambda) + \zeta_i(\lambda) r_{00}(\lambda) \eta_j(\lambda) \quad (i, \ j \geq 1). \quad (4.13)
\end{align*}
\]

Then \( R(\lambda) = \{r_{ij}(\lambda); \ i, j \in \mathbb{Z}_+\} \) constructed in this way is an honest \( Q \)-resolvent. Conversely, any honest \( Q \)-resolvent can be constructed in the above manner.

**Proof.** In the proof of Theorem 4.1, we have proved that \((4.1)\) has only the trivial solution. If we denote by \( \mathcal{M}^+(Q^*; \ H) \) the dimension of the solution space of \((4.1)\) (it is easily seen that this solution space is also independent of \( \lambda > 0 \)), as Chen and Renshaw (1993, Section 5) did (see \((5.4)\) there), then \( \mathcal{M}^+(Q^*; \ H) = 0 \). Now, using Theorem 6.1 of Chen and Renshaw (1993), we see that any honest \( Q \)-resolvent \( R(\lambda) \) can be represented as

\[
R(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \Phi^*(\lambda) \end{pmatrix} + r_{00}(\lambda) \begin{pmatrix} 1 \\ \zeta(\lambda) \end{pmatrix} (1 \ \eta(\lambda)).
\]

Here \( \Phi^*(\lambda) \) is the Feller minimal \( Q^* \)-resolvent. The terms \( r_{00}(\lambda) \) and \( \zeta(\lambda) \) should be in the forms \((4.10)\) and \((4.9)\) respectively, and

\[
\eta(\lambda) = H \Phi^*(\lambda) + \tilde{\eta}(\lambda),
\]

where \( \tilde{\eta}(\lambda) \) is a solution of the equation \((4.3)\) and satisfies the further condition that

\[
\tilde{\eta}(\lambda) - \tilde{\eta}(\mu) = (\mu - \lambda) \tilde{\eta}(\lambda) \Phi^*(\mu) \quad (\lambda, \ \mu > 0).
\]

However, by Remark 2.2, any such solution \( \tilde{\eta}(\lambda) \) must be in the form of \((2.32)\) and thus \( \eta(\lambda) \) must take the form of \((4.8)\). We have now proved that any honest \( Q \)-resolvent can be constructed as in \((4.8)-(4.13)\). The verification that each product of this construction is actually an honest \( Q \)-resolvent is easy and is contained in the proof of Theorem 6.1 of Chen and Renshaw (1993).
Remark 4.1. By Theorem 4.3, we see that the construction of all the honest $Q$-resolvents, when $Q$ is bilateral, is totally determined by four elements: the vector $H = \{h_j; j \geq 1\}$ given in (1.1); the Feller minimal $Q^*$-resolvent constructed in Lemma 2.4; $\tilde{W}(\lambda)$ in (2.32) and the nonnegative constant $c$. However, since the first three elements are totally determined by the BDDIR $q$-matrix $Q$ and are thus fixed, the construction of all the $Q$-resolvents is actually determined by the constant $c \geq 0$. Hence, all the $Q$-resolvents (and thus the $Q$-processes) can be indexed by the half line $T = [0, \infty)$. It is therefore reasonable to denote the set of all the $Q$-resolvents (when $Q$ is bilateral) as $\{R^c(\lambda); c \geq 0\}$. Note also that, when $Q$ is exit, the unique honest $Q$-resolvent constructed in Theorem 4.2 is exactly the same as $R^{(0)}(\lambda)$ here.

5. Ergodicity and equilibrium distribution

After establishing existence and uniqueness criteria, we now turn our attention to the properties of the BDDIR processes. In particular, we are interested in properties regarding recurrence, ergodicity and equilibrium distribution. Of course, we only need to consider the honest processes since the conclusions would be trivial for dishonest ones.

First consider the case when $Q$ is exit. By Theorem 4.1, there exists only one honest $Q$-process. We are now interested in finding conditions under which this unique honest process is ergodic. Interestingly, the following theorem shows that we do not need to impose any extra conditions.

Theorem 5.1. Suppose that the given BDDIR $q$-matrix $Q$ is exit. Then the unique honest $Q$-process is not only recurrent but also positive recurrent.

Proof. We only need to prove that this (unique) honest $Q$-process is always positive recurrent. By irreducibility and the Tauberian theorem, the $Q$-process is positive recurrent if and only if $\lim_{\lambda \to 0} \lambda r_{00}(\lambda) > 0$. In view of (4.4), this holds if and only if

$$
\lim_{\lambda \to 0} \sum_{k=1}^{\infty} h_k \sum_{j=1}^{\infty} \phi^*_k(j)(\lambda) < \infty.
$$

(5.1)

Since $\sum_{j=1}^{\infty} \phi^*_k(j)(\lambda)$ is a monotone function of $\lambda > 0$ for each $k$, using the monotone convergence theorem and (2.28) we see that (5.1) is true if and only if

$$
\sum_{k=1}^{\infty} h_k m_k < \infty,
$$

(5.2)

where $m_k, k \geq 1$, is given in (2.29). In order to finish the proof, we only need to show that (5.2) is always true. To this end, recall that there exists a BDDIR process if and only if (3.3) holds and hence it is sufficient to show that (3.3) implies (5.2). Since $\rho_n \geq 1$ given in (2.1) is increasing in $n$, by (2.12) we see that $u_n(0)$ given in (2.12) is also increasing in $n$. Using this fact it is easily seen that, for all $k \geq 1$, $m_k \leq g_k$ by comparing (2.4) and (2.5) with (2.29), and so (3.3) does imply (5.2).

Since any BDDIR $q$-matrix $Q$ is irreducible, the (unique) honest $Q$-process is ergodic and then possesses an equilibrium distribution. The following result shows exactly what this equilibrium distribution is.
Theorem 5.2. Suppose that the BDDIR $q$-matrix $Q$ is exit. Then the (unique) honest $Q$-process possesses an equilibrium distribution, denoted by $\Gamma^{(0)} = \{ \gamma_j^{(0)} ; \ j \geq 0 \}$, whose form is given by

$$\gamma_0^{(0)} = \frac{1}{1 + \sigma},$$

and, for $j \geq 1$,

$$\gamma_j^{(0)} = \pi_j \left[ u_j(0) \sum_{n=j}^{\infty} \frac{\rho_{n+1} - \rho_n}{u_{n+1}(0)u_n(0)} \sum_{k=1}^{j} h_k u_k(0) \right] + \sum_{k=j+1}^{\infty} h_k u_k(0) \sum_{n=k}^{\infty} \frac{\rho_{n+1} - \rho_n}{u_n(0)u_{n+1}(0)} \right] / (1 + \sigma),$$

where

$$\sigma = \sum_{k=1}^{\infty} h_k m_k$$

and $m_k$ for $k \geq 1$ is given in (2.29).

The reason for using the superscript in $\Gamma^{(0)}$ will become clear shortly.

Proof. Since the unique honest $Q$-process is both irreducible and positive recurrent, the equilibrium distribution $\Gamma^{(0)} = \{ \gamma_j^{(0)} ; \ j \geq 0 \}$ can be obtained by using

$$\gamma_j^{(0)} = \lim_{\lambda \to 0} \lambda r_{0j}(\lambda) \quad (j \geq 0),$$

where $r_{0j}(\lambda)$ for $j \geq 0$ is given in (4.4) and (4.5). Suppose that $j = 0$. Then, by virtue of (4.4), we know that

$$\lim_{\lambda \to 0} \lambda r_{00}(\lambda) = \left( 1 + \lim_{\lambda \to 0} \sum_{k=1}^{\infty} h_k \sum_{j=1}^{\infty} \phi_{kj}^*(\lambda) \right)^{-1},$$

which, by using the monotone convergence theorem and (2.28), immediately yields that

$$\lim_{\lambda \to 0} \lambda r_{00}(\lambda) = \left( 1 + \sum_{k=1}^{\infty} h_k m_k \right)^{-1}.$$

This is exactly $1/(1 + \sigma)$, where $\sigma$ is given in (5.5), and (5.3) is thus proved. Now, for $j \geq 1$, by using (4.5) and (5.3), we have

$$\gamma_j^{(0)} = \lim_{\lambda \to 0} \lambda r_{00}(\lambda) \eta_j(\lambda) = \lim_{\lambda \to 0} \left[ \sum_{k=1}^{\infty} h_k \phi_{kj}^*(\lambda) / (1 + \sigma) \right],$$

which, by using (2.26) and the monotone convergence theorem again, yields exactly (5.4).

We now consider the bilateral case. By Theorem 4.3, we know that in this situation there exist infinitely many honest $Q$-processes. At first sight, it seems that a complete clear-cut ergodicity solution would be very unlikely since such properties reasonably depend on the specific $Q$-process. Surprisingly, we may however prove the following theorem.
Theorem 5.3. Suppose that the given BDDIR q-matrix $Q$ is bilateral. Then each of the honest $Q$-processes is not only recurrent but also positive recurrent.

Proof. Again, we only need to prove the positive recurrence. By Theorem 4.3 and Remark 4.1, we know that there exists a one-to-one correspondence between the $Q$-process and the half line $[0, +\infty)$. For any $c \geq 0$, we denote by $R^{(c)}(\lambda) = \{r^{(c)}_{ij}(\lambda); \ i, j \in \mathbb{Z}_+\}$ the corresponding $Q$-resolvent. Then this specific $Q$-process is positive recurrent if and only if $\lim_{\lambda \to 0} \lambda r^{(c)}_{00}(\lambda) > 0$, which, by using (4.10), is true if and only if

$$\lim_{\lambda \to 0} (H \Phi^*(\lambda) + c \tilde{W}(\lambda) \mathbf{1}) < \infty. \tag{5.6}$$

The first term on the left-hand side of (5.6) is finite by the proof of Theorem 5.1. The second term is also finite. Indeed, as proved in Lemma 2.2 (see (2.19) there), the conditions $\pi_\sigma < \infty$ and $R < \infty$ imply that

$$\lim_{\lambda \to 0} c \tilde{W}(\lambda) \mathbf{1} = c \frac{1}{u_\infty(0)} \sum_{n=1}^{\infty} u_n(0) \pi_n < \infty. \tag{5.7}$$

The proof is complete.

Considering the fact that each of these $Q$-processes is ergodic, we are interested in finding the corresponding equilibrium distribution. We denote by $\Gamma^{(c)} = \{\gamma_j^{(c)}; \ j \geq 0\}$ the specific equilibrium distribution corresponding to the $Q$-process indexed by $c \geq 0$.

Theorem 5.4. For any $c \geq 0$, the equilibrium distribution $\Gamma^{(c)} = \{\gamma_j^{(c)}; \ j \geq 0\}$ is given by

$$\gamma_0^{(c)} = \frac{1}{1 + \sigma + c(1/u_\infty(0)) \sum_{n=1}^{\infty} u_n(0) \pi_n} \tag{5.8}$$

and, for $j \geq 1$,

$$\gamma_j^{(c)} = \frac{(1 + \sigma)\gamma_j^{(0)} + c(1/u_\infty(0))u_j(0)\pi_j}{1 + \sigma + c(1/u_\infty(0)) \sum_{n=1}^{\infty} u_n(0) \pi_n}, \tag{5.9}$$

where $\sigma$ and $\gamma_j^{(0)}$ for $j \geq 1$ are given in (5.5) and (5.4) respectively.

Proof. By using (4.10), (2.19) and (5.3), we have

$$\gamma_0^{(c)} = \lim_{\lambda \to 0} \lambda r^{(c)}_{00}(\lambda) = \frac{1}{1 + \sigma + c(1/u_\infty(0)) \sum_{n=1}^{\infty} u_n(0) \pi_n},$$

and thus (5.8) is proved. Furthermore, for $j \geq 1$, by using (4.8) and (4.11), we obtain that

$$\gamma_j^{(c)} = \lim_{\lambda \to 0} \lambda r^{(c)}_{0j}(\lambda) = \lim_{\lambda \to 0} \lambda r^{(c)}_{00}(\lambda) \left[ \sum_{k=1}^{\infty} h_k \Phi^*_k(\lambda) + c \tilde{W}_j(\lambda) \right],$$

which, by using (5.4), (5.8) and (2.17), is equal to (5.9).

By Theorem 5.4, we see that the structure of the equilibrium distributions is very simple. It is totally determined by the nonnegative constant $c$ and the equilibrium distribution $\Gamma^{(0)}$, which is given in Theorem 5.2. Also, when $c = 0$, $\Gamma^{(0)}$ is as given in Theorem 5.2. This is the reason that we used the notation $\Gamma^{(0)}$ in Theorem 5.2.
6. Symmetry and reversibility

We now turn our attention to another very important question. For a given BDDIR pre-\(q\)-matrix \(Q\), under what conditions does there exist an honest weakly symmetric or reversible \(Q\)-process? Furthermore, under what conditions does there exist only one such process? Again, we shall mainly consider this question under the assumption that \(R < \infty\).

We refer the reader to Anderson (1991) and M. F. Chen (1992) for the general concepts and basic conclusions as well as the terminology and notation regarding weak symmetry and reversibility. In particular, recall that a transition function \(P(t) = \{p_{ij}(t)\; i, j \in E\}\), where the state space \(E\) is a countable set, is called weakly symmetric if there exists a set \(\{\mu_i; \; i \in E\}\) of strictly positive numbers such that \(\mu_i p_{ij}(t) = \mu_j p_{ji}(t)\) for all \(i, j \in E\) and \(t \geq 0\). If we further have \(\sum_{i \in E} \mu_i < +\infty\), then \(P(t)\) is called symmetric. In both cases, this set \(\{\mu_i; \; i \in E\}\) is called a symmetrizing measure.

We now begin to consider our BDDIR reversible processes. The first thing we need to do is to recognize the symmetrizing measure for our BDDIR pre-\(q\)-matrix \(Q\).

**Theorem 6.1.** Let \(Q\) be a BDDIR pre-\(q\)-matrix. Then \(Q\) is weakly symmetric if and only if

\[
h_1 > 0 \quad \text{and} \quad (a_1 + d_1)h_j = h_j \pi_j d_j \quad \text{for} \; j \geq 2.
\]

(6.1)

If (6.1) is true, then all the symmetrizing measures can be expressed as \(\mu_0 = c(a_1 + d_1)/h_1\) and \(\mu_j = c\pi_j\) (for \(j \geq 1\)), where \(c > 0\) is a constant. Furthermore, \(Q\) is symmetric if and only if (6.1) and \(\pi_\sigma < \infty\) hold.

**Proof.** If \(Q\) is weakly symmetric, then there exists a symmetrizing measure \(\{\mu_i; \; i \in \mathbb{Z}_+\}\), say, such that

\[
\mu_0 h_j = \mu_j (d_j + \delta_{1j} a_1) \quad (j \geq 1)
\]

(6.2)

and

\[
\mu_i q^*_j = \mu_j q^*_j \quad (i, j \in \mathbb{Z}_{++}),
\]

(6.3)

where \(Q^* = \{q^*_j\}\) is the restriction of \(Q\) to \(\mathbb{Z}_{++}\), as defined in the beginning of Section 2. Since \(Q^*\) is an ordinary (nonconservative) birth–death \(q\)-matrix which already possesses a symmetrizing measure \(\{\pi_i; \; i \geq 1\}\) and \(Q^*\) is irreducible, by (6.3) we know that there exists a constant \(c > 0\) such that

\[
\mu_j = c\pi_j \quad (j \geq 1).
\]

(6.4)

Now, letting \(j = 1\) in (6.2), we first get \(\mu_0 h_1 = \mu_1 (a_1 + d_1)\) and thus \(h_1 > 0\) and also \(\mu_0 h_1 = c\pi_1 (a_1 + d_1) = c(a_1 + d_1)\) as \(\pi_1 = 1\). Moreover, for \(j \geq 2\), (6.2) yields \(\mu_0 h_j = \mu_j d_j\) or, equivalently,

\[
\frac{c(a_1 + d_1)}{h_1} h_j = c\pi_j d_j \quad (j \geq 2).
\]

Thus, (6.1) follows. Conversely, if (6.1) is true, then let \(\mu_0 = (a_1 + d_1)/h_1\) and \(\mu_j = \pi_j\) (for \(j \geq 1\)) it is easily checked that \(\mu_j > 0\) (for \(j \geq 0\)) and \(\{\mu_j; \; j \geq 0\}\) defined in this way is indeed a symmetrizing measure with \(Q\). Thus, all the symmetrizing measures can be given as stated above. The last statement is obvious.

Note that (6.1) and (1.2) imply that \(\sum_{j=1}^{\infty} \pi_j d_j = +\infty\). Note also that Theorem 6.1 does not depend on \(R < \infty\). We now assume that \(R < \infty\) holds and investigate the existence of honest weakly symmetric processes. Again, we shall consider two cases, bilateral and exit, separately.
Theorem 6.2. Suppose that the given BDDIR pre-q-matrix is exit (i.e. $R < \infty$ and $\pi_\sigma = \infty$). Then there are no honest weakly symmetric processes.

Proof. Suppose that there exists an honest weakly symmetric process with a symmetrizing measure $\mu = \{\mu_j; j \geq 0\}$. Then $\sum_{i=0}^{\infty} \mu_i = +\infty$ since there exists a constant $c > 0$ such that (6.4) holds. Let $R(\lambda) = \{r_{ij}(\lambda); i, j \geq 0\}$ be the corresponding honest weakly symmetric resolvent with the symmetrizing measure $\mu_\sigma$. Then this $R(\lambda)$, being an honest $Q$-resolvent, must have entries of the form (4.4)–(4.7) in Theorem 4.2. Now, since $R(\lambda)$ is weakly symmetric with $\mu$, we must have

$$\mu_0 r_{0j}(\lambda) = \mu_j r_{j0}(\lambda) \quad (j \geq 1). \quad (6.5)$$

Substituting (4.5) and (4.6) into (6.5) immediately yields that

$$\mu_0 \sum_{k=1}^{\infty} h_k \phi_{kj}^*(\lambda) = \mu_j \left(1 - \lambda \sum_{k=1}^{\infty} \phi_{jk}^*(\lambda)\right) \quad (j \geq 1). \quad (6.6)$$

However, by the existence conditions (3.2) and (6.6),

$$\sum_{j=1}^{\infty} \mu_j \left(1 - \lambda \sum_{k=1}^{\infty} \phi_{jk}^*(\lambda)\right) < +\infty,$$

which, by using (2.25), implies that

$$\sum_{j=1}^{\infty} \mu_j u_j(\lambda) \frac{1}{u_\infty(\lambda)} < +\infty. \quad (6.7)$$

But $1/u_\infty(\lambda) > 0$, $u_j(\lambda) \geq u_1(\lambda) = 1$ (for $j \geq 1$) and hence (6.7) implies that $\sum_{j=1}^{\infty} \mu_j < +\infty$, which is a contradiction.

At first sight, Theorem 6.2 seems a little peculiar since the BDDIR q-matrix $Q$ can be weakly symmetric, as shown in Theorem 6.1, but there are no honest weakly symmetric processes. However, if we recall Theorem 5.1 (positive recurrence), this would be obvious.

Finally, we consider the bilateral case. Of course, for such a q-matrix $Q$, there are no nonsummable weakly symmetric $Q$-processes. So, we only need to consider the existence of symmetric (and thus summable) $Q$-processes. By normalizing the symmetric measure, we may, without loss of generality, only consider the reversible $Q$-process. The following is the main result of this section.

Theorem 6.3. Suppose that $Q$ is bilateral. Then there exists a reversible $Q$-process if and only if (6.1) holds. Moreover, if (6.1) does hold, then the reversible $Q$-process is unique. This unique reversible $Q$-resolvent can be constructed as follows: let $c = h_1(a_1 + d_1)^{-1}$ in (4.8) and then construct the $Q$-resolvent as in (4.8)–(4.13). The equilibrium distribution of this reversible $Q$-process, $\Gamma = \{\gamma_j; j \geq 0\}$, is given by

$$\gamma_j = \frac{\pi_j}{\sum_{i=0}^{\infty} \pi_i}, \quad (6.8)$$

where $\pi_0 = (a_1 + d_1)h_1^{-1}$ and $\pi_j$ for $j \geq 1$ is given in (2.2).
Proof. The necessity of (6.1) is obvious. We now prove that (6.1) is also sufficient and, furthermore, that if (6.1) holds, there exists only one reversible process. Recall that (see M. F. Chen (1992)) a $Q$-process is reversible if and only if it is honest, irreducible and symmetric. Since any honest $Q$-resolvent can be constructed as in (4.8)–(4.13) and each of these honest $Q$-resolvents is obviously irreducible, we only need to prove that, if (6.1) holds, then there does exist a unique $Q$-resolvent which is symmetric with, for instance, $\{\pi_i; i \geq 0\}$, where $
_0 = (a_1 + d_1)h_1^{-1}$. Now, using (4.8)–(4.13), it is easy to see that any such $Q$-resolvent is symmetric with $\{\pi_i; i \geq 0\}$ if and only if both

$$\pi_0 \eta_j(\lambda) = \pi_j \xi_j(\lambda) \quad (j \geq 1) \quad (6.9)$$

and

$$\pi_i r_{ij}(\lambda) = \pi_j r_{ji}(\lambda) \quad (i, j \geq 1) \quad (6.10)$$

hold, where $\{\eta_j(\lambda); j \geq 1\}$, $\{\xi_j(\lambda); j \geq 1\}$ and $\{r_{ij}(\lambda); i, j \geq 1\}$ are given in (4.8), (4.9) and (4.13) respectively. Substituting (4.13) into (6.10) and using the fact that $\pi_i \phi_{ij}^*(\lambda) = \pi_j \phi_{ji}^*(\lambda)$ (see (2.30)) yield the fact that (6.10) holds if and only if

$$\pi_i \xi_i(\lambda) \eta_j(\lambda) = \pi_j \xi_j(\lambda) \eta_i(\lambda) \quad (i, j \geq 1),$$

which can be deduced from (6.9). Hence, it suffices to consider (6.9) only. By substituting (4.8) into (6.9) and using (2.25), we see that (6.9) is true if and only if

$$\pi_0 \left[ \sum_{k=1}^{\infty} h_k \phi_{kj}^*(\lambda) + c \tilde{w}_j(\lambda) \right] = \pi_j \left[ a_1 v_j(\lambda) + \sum_{k=1}^{\infty} \phi_{jk}^*(\lambda) d_k + u_j(\lambda) \frac{1}{u_\infty(\lambda)} \right]. \quad (6.11)$$

However, using the condition (6.1) and the fact that $v_j(\lambda) = \phi_{jj}^*(\lambda)$, we see that

$$\pi_j \left[ a_1 \phi_{jj}^*(\lambda) + \sum_{k=1}^{\infty} \phi_{jk}^*(\lambda) d_k \right] = (a_1 + d_1) \pi_1 \phi_{jj}^*(\lambda) + \sum_{k=2}^{\infty} \phi_{kj}^*(\lambda) \pi_k d_k$$

$$= (a_1 + d_1) \phi_{jj}^*(\lambda) + \sum_{k=2}^{\infty} (a_1 + d_1) h_k h_1^{-1} \phi_{kj}^*(\lambda)$$

$$= \pi_0 \left[ \sum_{k=1}^{\infty} h_k \phi_{kj}^*(\lambda) \right].$$

Thus (6.11) is true, i.e. there exists a reversible $Q$-process if and only if

$$c \pi_0 \tilde{w}_j(\lambda) = \pi_j u_j(\lambda) \frac{1}{u_\infty(\lambda)},$$

which, by using the fact that $\tilde{w}_j(\lambda) = \pi_j u_j(\lambda)(1/u_\infty(\lambda))$ (see (2.32) and (2.16)), holds if and only if $c = \pi_0^{-1} = h_1(a_1 + d_1)^{-1}$. Now all the conclusions, including the existence and uniqueness of reversible $Q$-processes, follow. Hence, the proof is complete.
7. Examples

In this section, we study some examples to illustrate the results derived in the previous sections.

**Example 7.1.** Suppose that the BDDIR pre-q-matrix $Q$ takes the following form: for $n \geq 1$,

$$b_n = b n^\theta, \quad a_n = a n^\theta, \quad d_n = d n^\tau$$

with $a > 0$, $b > 0$, $d \geq 0$, \hspace{1cm} (7.1)

together with a general form of $h_n$ as in (1.2), where $\theta$ and $\tau$ are two real numbers.

Some algebra shows that the criterion (3.3) can now be written as the following two conditions:

$$\sum_{k=1}^{\infty} k^{-\theta} \left( \sum_{i=k}^{\infty} h_i \left( \frac{a}{b} \right)^{i-k} \right) < \infty$$

and

$$\sum_{k=2}^{\infty} \sum_{i=1}^{k-1} k^{-\theta} h_i < \infty. \hspace{1cm} (7.3)$$

Applying the results obtained before, we get the following conclusion.

**Theorem 7.1.** For the BDDIR pre-q-matrix $Q$ defined in (1.1), (1.2) and (7.1), $R < \infty$ if and only if $\theta > \max \{ \tau + 1, 1 \}$ and $b > a$. Under the latter conditions, there exists a $Q$-process (and therefore an honest $Q$-process) if and only if (7.2) and (7.3) hold. When these conditions are satisfied, the honest $Q$-process is unique and ergodic. The equilibrium distribution $\Gamma = \{ y_j : j \geq 0 \}$ may be obtained by using (5.3)-(5.5). Moreover, there are no honest weakly symmetric $Q$-processes.

**Example 7.2.** This example is a special case of Example 7.1. Let the BDDIR pre-q-matrix $Q$ take the form of (7.1), together with

$$h_n = h n^\delta \quad \text{for } n \geq 1 \text{ with } h > 0, \delta \geq -1. \hspace{1cm} (7.4)$$

Note that we require $\delta \geq -1$ in (7.4) since we need to make sure that (1.2) is true. The pre-q-matrix with the special form $\delta = 0$ (and thus $h_n = h > 0$ in (7.4)) is called the Kolmogorov–Williams-type pre-q-matrix. Using our previous results, together with some algebra, we may come to the following interesting conclusion.

**Theorem 7.2.** Suppose that $Q$ is a BDDIR pre-q-matrix as in (7.1) and (7.4). Then $R < \infty$ if and only if $\theta > \max \{ \tau + 1, 1 \}$ and $b > a$. If these conditions are satisfied, then there exists a $Q$-process (and therefore an honest $Q$-process) if and only if the following further condition holds:

$$\theta > \delta + 2. \hspace{1cm} (7.5)$$

Furthermore, if (7.5) is true, then the honest $Q$-process is unique, recurrent and ergodic. In particular, if $h_n \equiv 1$ (for $n \geq 1$), i.e. $\delta = 0$, $h = 1$, then there exists a $Q$-process (and therefore an honest $Q$-process) if and only if $\theta > 2$ and the honest $Q$-process is unique, recurrent and ergodic.
**Example 7.3.** In Examples 7.1 and 7.2, the given BDDIR $q$-matrices are exit. We now give another family of examples whose BDDIR $q$-matrices are bilateral. Let the BDDIR pre-$q$-matrix take the form

$$a_n = n^{2\theta}, \quad b_n = n^{\theta}(n + 1)^{\theta}, \quad d_n = dn^\tau \quad \text{for } n \geq 1, d \geq 0, \quad (7.6)$$

together with the general form of $h_n$ as in (1.2), where $\theta$ and $\tau$ are two real numbers.

For this example, it is easy to see that $R < \infty$ if and only if $\theta > \max\{\tau/2 + 1, 1\}$ and, under this condition, the given BDDIR pre-$q$-matrix becomes a $q$-matrix if and only if

$$\sum_{n=2}^{\infty} n^{-\theta} \left( \sum_{j=1}^{n-1} h_j \right) < \infty. \quad (7.7)$$

We may now obtain the following conclusion.

**Theorem 7.3.** For the BDDIR pre-$q$-matrix $Q$ defined in (1.1), (1.2) and (7.6), if $\theta > \max\{\tau/2 + 1, 1\}$, then there exists a $Q$-process (i.e. $Q$ becomes a BDDIR $q$-matrix) if and only if (7.7) holds. Moreover, there exists a constant $\theta_0 > 1$ such that, if $\theta > \max\{\tau/2 + 1, \theta_0\}$, then $Q$ is a $q$-matrix while, if $\max\{\tau/2 + 1, 1\} < \theta < \theta_0$, then $Q$ is not a $q$-matrix. Furthermore, if the above existence conditions are satisfied, then there exist infinitely many honest $Q$-processes which can be indexed by the half line $T = [0, \infty)$. If the sequence $\{h_j; \ j \geq 1\}$ satisfies the conditions $\inf_{i \geq 1} h_i > 0$, $\sup_{i \geq 1} h_i < \infty$ and $\theta > \max\{\tau/2 + 1, 1\}$, then there exists a $Q$-process if and only if $\theta > 2$. In particular, if $h_j \equiv 1$ (for $j \geq 1$) and $\theta > \max\{\tau/2 + 1, 1\}$, then there exists a $Q$-process if and only if $\theta > 2$.

**Example 7.4.** As the last example, we consider the sequences $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ given by

$$b_n = bl^n, \quad a_n = al^n, \quad d_n = dl^{n/2} \quad \text{with } a > 0, \ b > 0, \ d \geq 0, \ l > 0, \quad (7.8)$$

together with the general form of $h_n$ as in (1.1). Using the previous results, we obtain the following conclusion.

**Theorem 7.4.** Suppose that $Q$ is a BDDIR pre-$q$-matrix as in (7.8) and (1.1).

(i) If $a < b < al$, then there exists a $Q$-process (and therefore an honest $Q$-process) if and only if

$$\sum_{k=1}^{\infty} h_k \left( \frac{a}{b} \right)^k < \infty,$$

in which case there exist infinitely many honest $Q$-processes.

(ii) If $b = al > a$, then there exists a $Q$-process (and therefore an honest $Q$-process) if and only if

$$\sum_{k=1}^{\infty} kh_k l^{-k} < \infty,$$

in which case there exists only one honest $Q$-process.
(iii) If \( b > al > a \), then there exists a \( Q \)-process (and therefore an honest \( Q \)-process) if and only if
\[
\sum_{k=1}^{\infty} h_k l^{-k} < \infty,
\]
in which case there exists only one honest \( Q \)-process.

(iv) If \( l > 1 \) and \( a < b \), then there exists a reversible \( Q \)-process if and only if
\[
h_1(al + dl^{1/2})^{-1} = h_n d^{-1} \left( \frac{a}{b} \right)^{n-1} l^{n/2-1}, \quad n \geq 2,
\]
and
\[
al > b \geq al^{1/2},
\]
in which case there exists a unique reversible \( Q \)-process. Its equilibrium distribution is given by
\[
y_0 = \frac{al - b}{al - b + alc} ,
y_n = \frac{(al - b)c}{al - b + alc} \left( \frac{b}{al} \right)^{n-1}, \quad n \geq 1,
\]
where \( c = h_1(al + dl^{1/2})^{-1} \).

**Appendix A. Proofs of Lemmas 2.1–2.4**

In this last section, we shall prove the basic lemmas stated in Section 2 regarding nonconservative birth–death processes.

**A.1. Proof of Lemma 2.1**

Substituting the form of \( Q^* \) into (2.10) immediately yields that
\[
\quad u_2(\lambda) = \frac{\lambda + a_1 + b_1 + d_1}{b_1} u_1(\lambda)
\]
and, for \( n \geq 2 \),
\[
\quad u_{n+1}(\lambda) = \frac{\lambda + a_n + b_n + d_n}{b_n} u_n(\lambda) - \frac{a_n}{b_n} u_{n-1}(\lambda).
\]
Letting \( u_1(\lambda) = 1 \) and applying the induction principle and a little algebra yield (2.11). Moreover, by (A.1) and (A.2) it is easily seen that each solution of (2.10) can be expressed as \( \{c(\lambda)u_n(\lambda) ; n \geq 1\} \). Note also that, by (A.1) and (A.2), we have that
\[
u_2(\lambda) - u_1(\lambda) = \frac{\lambda + a_1 + d_1}{b_1} u_1(\lambda),
\]
\[
u_{n+1}(\lambda) - u_n(\lambda) = \frac{\lambda + d_n}{b_n} u_n(\lambda) + \frac{a_n}{b_n} (u_n(\lambda) - u_{n-1}(\lambda)) \quad (\text{for } n \geq 2).
\]
Therefore, for each \( \lambda > 0 \), \( u_n(\lambda) \) is a strictly increasing function of \( n \) and thus the limit \( \lim_{n \to \infty} u_n(\lambda) := u_\infty(\lambda) \leq +\infty \) exists. Applying Reuter’s lemma (see Reuter (1957) or
Anderson (1991, Lemma 3.2.1, p. 98)) in (A.3) and (A.4) directly yields the fact that \( u_n(\lambda) \) is bounded or, equivalently, \( u_\infty(\lambda) < \infty \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{\lambda + d_n}{b_n} + \frac{a_n(\lambda + d_{n-1})}{b_nb_{n-1}} + \frac{a_na_{n-1}(\lambda + d_{n-2})}{b_nb_{n-1}b_{n-2}} + \ldots \right.
\]

\[
+ \left. \frac{a_na_{n-1} \ldots a_2(\lambda + a_1 + d_1)}{b_nb_{n-1}b_{n-2} \ldots b_2b_1} \right) < +\infty. \tag{A.5}
\]

However, the condition (A.5) is independent of \( \lambda > 0 \) and thus we may take \( \lambda = 1 \). Also, since \( a_1 > 0 \), it is easily seen that (A.5) is equivalent to the condition \( R < \infty \), where \( R \) is defined in (2.9). This completes the proof of (i) in Lemma 2.1.

Conclusion (ii) in Lemma 2.1 follows directly from (2.11) and the proof of (iii) is immediate. Indeed, if \( R < \infty \), then we may write the finite-valued function \( u_\infty(\lambda) \) as

\[
u_\infty(\lambda) = \sum_{n=2}^{\infty} (u_n(\lambda) - u_{n-1}(\lambda)) + u_1(\lambda). \tag{A.6}
\]

Now by (A.4) it is easily seen that, for all \( n \geq 2 \), \( u_n(\lambda) - u_{n-1}(\lambda) \) is a positively increasing function of \( \lambda > 0 \), and thus so is \( u_\infty(\lambda) \) by (A.6). Applying the monotone convergence theorem on the right-hand side of (A.6) yields (2.13). Finally, (iv) is obvious and this completes the proof of Lemma 2.1.

A.2. Proof of Lemma 2.2

Substituting the form of \( Q^* \) into the equation \( W(\lambda)(\lambda I - Q^*) = 0 \) immediately yields that

\[
w_2(\lambda) = \frac{\lambda + a_1 + b_1 + d_1}{a_2} w_1(\lambda) \tag{A.7}
\]

and

\[
w_{n+1}(\lambda) = \frac{\lambda + a_n + b_n + d_n}{a_{n+1}} w_n(\lambda) - \frac{b_{n-1}}{a_{n+1}} w_{n-1}(\lambda) \quad (n \geq 2). \tag{A.8}
\]

We now prove that, for all \( n \geq 1 \), we actually have that

\[
w_n(\lambda) = u_n(\lambda)\pi_n. \tag{A.9}
\]

For \( n = 1 \), (A.9) is trivially true since \( w_1(\lambda) = u_1(\lambda) = \pi_1 = 1 \). Then, for \( n = 2 \), by using (A.7) we obtain

\[
w_2(\lambda) = \frac{\lambda + a_1 + b_1 + d_1}{a_2} u_1(\lambda)\pi_1 = \frac{\lambda + a_1 + b_1 + d_1}{b_1} u_1(\lambda)\frac{b_1}{a_2}\pi_1 = u_2(\lambda)\pi_2,
\]

and so (A.9) also holds for \( n = 2 \). Now suppose that (A.9) holds for all positive integers up to \( n \). Then, for \( n + 1 \), by using (A.8) we have that

\[
w_{n+1}(\lambda) = \frac{\lambda + a_n + b_n + d_n}{a_{n+1}} u_n(\lambda)\pi_n - \frac{b_{n-1}}{a_{n+1}} u_{n-1}(\lambda)\pi_{n-1}
\]

\[
= \frac{\lambda + a_n + b_n + d_n}{b_n} u_n(\lambda)\frac{b_n}{a_{n+1}}\pi_n - \frac{a_n}{b_n} u_{n-1}(\lambda)\frac{b_n}{a_{n+1}}\pi_{n-1}
\]

\[
= \left( \frac{\lambda + a_n + b_n + d_n}{b_n} u_n(\lambda) - \frac{a_n}{b_n} u_{n-1}(\lambda) \right)\pi_{n+1},
\]
which, by using (A.2), is exactly $u_{n+1}(\lambda)\pi_{n+1}$. Hence, (A.9) holds for all $n \geq 1$. The conclusion (i) in Lemma 2.2 now follows from (A.9) and Lemma 2.1. Conclusion (ii) in Lemma 2.2 is also an immediate consequence of the facts just proven. That is, that $\mathcal{N}^+(Q^*)$ can only be either 1 or 0 and that $\mathcal{N}^+(Q^*)$ is 1 or 0 according to whether $\sum_{n=1}^{\infty} u_n(\lambda)\pi_n$ is convergent or not. However, $R < \infty$ implies that $u_\infty(\lambda) < \infty$ and thus

$$
\sum_{n=1}^{\infty} \pi_n \leq \sum_{n=1}^{\infty} u_n(\lambda)\pi_n \leq u_\infty(\lambda) \sum_{n=1}^{\infty} \pi_n,
$$

which shows that $\mathcal{N}^+(Q^*)$ is 1 or 0 depending on whether $\sum_{n=1}^{\infty} \pi_n$ is convergent or not. This finishes the proof of (iii) in Lemma 2.2. Finally, the conclusion (iv) in Lemma 2.2, particularly (2.19), is immediate and thus the lemma is proved.

A.3. Proof of Lemma 2.3

We introduce an operator $\ast$ as follows. Suppose that $U = (u_i; \ i \in \mathbb{Z}^+)$ is a column vector on $\mathbb{Z}^+$. Define $U^+ = (u_i^+; \ i \in \mathbb{Z}^+)$ by

$$
u^+_i = \frac{u_i+1 - u_i}{\rho_{i+1} - \rho_i} \quad (i \geq 1). \tag{A.10}
$$

By (2.11) and the operator (A.10), it is easily seen that

$$u^+_n(\lambda) = a_1 + \sum_{j=1}^{n} (\lambda + d_j)u_j(\lambda)\pi_j; \tag{A.11}
$$

hence, $u^+_n(\lambda)$ is strictly increasing in $n$ for all $\lambda > 0$. Thus, by (2.20) we have that

$$
0 < v_n(\lambda) = u_n(\lambda) \sum_{j=n}^{\infty} \frac{1}{u_j^+(\lambda)} \left( \frac{1}{u_j(\lambda)} - \frac{1}{u_{j+1}(\lambda)} \right)
\leq u_n(\lambda) \left( \frac{1}{u_n(\lambda)} - \frac{1}{u_\infty(\lambda)} \right) \leq \frac{1}{u_\infty(\lambda)} < \infty. \tag{A.12}
$$

Therefore, the series on the right-hand side of (2.20) is convergent and hence (i) in Lemma 2.3 is proved.

In order to prove (ii), first note that we have the trivial equality

$$
0 \leq v_n(\lambda) = u_n(\lambda) \sum_{j=n}^{\infty} \frac{1}{u_j^+(\lambda)} \left( \frac{1}{u_j(\lambda)} - \frac{1}{u_{j+1}(\lambda)} \right) = 1, \quad n \geq 1. \tag{A.13}
$$

Now, using (A.12) and (A.13), we have that

$$
v_n^+(\lambda) < \frac{1}{u_n(\lambda)} - \frac{1}{u_n(\lambda)} = 0;
$$

hence, $v_n(\lambda)$ is strictly decreasing in $n$.

Secondly, by (A.13), (A.11) and the increasing property of $u^+_n(\lambda)$ with $n$, we obtain that

$$
-\frac{1}{u_n(\lambda)} \leq v_n^+(\lambda) \leq \sum_{j=n}^{\infty} u_j^+(\lambda) - \frac{\rho_{j+1} - \rho_j}{u_j(\lambda)u_{j+1}(\lambda)} \leq \sum_{j=n}^{\infty} \left( \frac{1}{u_j(\lambda)} - \frac{1}{u_{j+1}(\lambda)} \right) \leq \frac{1}{u_\infty(\lambda)}. \tag{A.14}
$$
which immediately yields (2.21). Now we prove that

\[
(\lambda + \tilde{d}_n)v_n(\lambda) - \frac{v_n^+(\lambda) - v_{n-1}^+(\lambda)}{\pi_n} = \begin{cases} 
0 & \text{if } n \geq 2, \\
1 & \text{if } n = 1, 
\end{cases}
\]

(A.14)

where \( \tilde{d}_n = d_n \) (for \( n \geq 2 \)), \( \tilde{d}_1 = a_1 + d_1 \) and \( v_0^+(\lambda) = 0. \) Indeed, for \( n \geq 2 \), by using (A.13) and (A.11), we get that

\[
\frac{v_n^+(\lambda) - v_{n-1}^+(\lambda)}{\pi_n} = \frac{1}{\pi_n} \left( (u_n^+(\lambda) - u_{n-1}^+(\lambda)) \sum_{j=n}^{\infty} \frac{\rho_j+1 - \rho_j}{u_j(\lambda)u_{j+1}(\lambda)} + \frac{u_n(\lambda) - u_{n-1}(\lambda)}{u_{n-1}(\lambda)u_n(\lambda)} 
- u_{n-1}^+(\lambda) \frac{\rho_n - \rho_{n-1}}{u_{n-1}(\lambda)u_n(\lambda)} \right)
= (\lambda + d_n)u_n(\lambda) \sum_{j=n}^{\infty} \frac{\rho_j+1 - \rho_j}{u_j(\lambda)u_{j+1}(\lambda)}
= (\lambda + d_n)v_n(\lambda),
\]

while, for \( n = 1 \), we have that

\[
(\lambda + \tilde{d}_1)v_1(\lambda) - \frac{v_1^+(\lambda) - v_0^+(\lambda)}{\pi_1} = (\lambda + a_1 + d_1 - u_1^+(\lambda)) \sum_{j=1}^{\infty} \frac{\rho_j+1 - \rho_j}{u_j(\lambda)u_{j+1}(\lambda)} + 1 = 1.
\]

Thus (A.14) is proved. Now, by noting that

\[
a_n(\rho_n - \rho_{n-1}) = b_n(\rho_{n+1} - \rho_n) = \frac{1}{\pi_n},
\]

we recognize that (A.14) is exactly (2.23) and (2.22) for \( n \geq 2 \) and \( n = 1 \) respectively.

A.4. Proof of Lemma 2.4

Our last task is to prove the main construction theorem, Lemma 2.4. In order to do this, we first prove the following propositions.

Proposition A.1. The relation (2.25) holds.

Proof. By using (A.11), (A.13), (A.14) and (2.21), we have that

\[
\lambda \sum_{j=1}^{\infty} \phi^*_{ij}(\lambda) + \sum_{j=1}^{\infty} \phi^*_{ij}(\lambda)d_j = v_i(\lambda) \sum_{j=1}^{i} (\lambda + d_j)u_j(\lambda)\pi_j + u_i(\lambda) \sum_{j=i+1}^{\infty} (\lambda + d_j)v_j(\lambda)\pi_j
= v_i(\lambda)(u_i^+(\lambda) - a_1) + u_i(\lambda) \sum_{j=i+1}^{\infty} (v_j^+(\lambda) - v_{j-1}^+(\lambda)),
\]

which, by using (A.13) and (2.21), yields (2.25).

In the following two propositions, let \( \mathcal{M} \) and \( \mathcal{L} \) denote the spaces of bounded column vectors and absolutely summable row vectors on \( \mathbb{Z}_{++} \) respectively. Suppose that \( f \) is a column vector.
on $\mathbb{Z}_{++}$ and $g$ is a row vector on $\mathbb{Z}_{++}$. Then by (2.24) we have

$$
[\Phi^*(\lambda)f]_i = \sum_{j=1}^{\infty} \phi^*_j(\lambda)f_j = v_i(\lambda)\sum_{j=1}^{i} u_j(\lambda)\pi_j f_j + u_i(\lambda)\sum_{j=i+1}^{\infty} v_j(\lambda)\pi_j f_j, 
$$

(A.16)

$$
[g\Phi^*(\lambda)]_j = \sum_{i=1}^{\infty} g_i\phi^*_j(\lambda) = v_j(\lambda)\pi_j \sum_{i=1}^{j} g_iu_i(\lambda) + u_j(\lambda)\pi_j \sum_{i=j+1}^{\infty} g_i v_i(\lambda).
$$

(A.17)

**Proposition A.2.** If $f \in \mathcal{M}$ and $g \in \mathcal{L}$, then $\Phi^*(\lambda)f \in \mathcal{M}$, $g\Phi^*(\lambda) \in \mathcal{L}$ and

$$
\lambda[\Phi^*(\lambda)f] - Q^*[\Phi^*(\lambda)f] = f, \quad \lambda > 0,
$$

(A.18)

$$
\lambda[g\Phi^*(\lambda)] - [g\Phi^*(\lambda)]Q^* = g, \quad \lambda > 0.
$$

(A.19)

**Proof.** By the proven (2.25) (see Proposition A.1) and using (A.16) and (A.17), it is easily seen that, if $f \in \mathcal{M}$, then $\Phi^*(\lambda)f \in \mathcal{M}$ and, if $g \in \mathcal{L}$, then $g\Phi^*(\lambda) \in \mathcal{L}$ for each $\lambda > 0$. Now we prove (A.18). For $i = 1$, by using (2.10), (2.24), (2.20) and (A.15), we have that

$$
\lambda(\Phi^*(\lambda)f)_1 - (Q^*[\Phi^*(\lambda)f])_1
$$

$$
= (\lambda u_1(\lambda) + (d_1 + a_1 + b_1)u_1(\lambda) - b_1u_2(\lambda))\sum_{j=1}^{\infty} v_j(\lambda)\pi_j f_j
$$

$$
- b_1v_2(\lambda)u_1(\lambda)\pi_1 f_1 + b_1u_2(\lambda)v_1(\lambda)\pi_1 f_1
$$

$$
= b_1\pi_1 f_1(v_1(\lambda)u_2(\lambda) - v_2(\lambda)u_1(\lambda))
$$

$$
= f_1\pi_1 b_1u_1(\lambda)u_2(\lambda)\frac{\rho_2 - \rho_1}{u_1(\lambda)u_2(\lambda)} = f_1,
$$

so (A.18) holds for $i = 1$. For $i \geq 2$, again by using (2.10), (2.24), (2.20) and (A.15), we get that

$$
\lambda[\Phi^*(\lambda)f]_i - [Q^*(\Phi^*(\lambda)f)]_i = \lambda\sum_{j=1}^{\infty} \phi^*_j(\lambda)f_j - a_i\sum_{j=1}^{\infty} \phi^*_{i-1,j}(\lambda)f_j
$$

$$
+ (a_i + b_i + d_i)\sum_{j=1}^{i} \phi^*_j(\lambda)f_j - b_i\sum_{j=1}^{\infty} \phi^*_{i+1,j}(\lambda)f_j,
$$

which, for convenience, is written as $I_1 + I_2$, where

$$
I_1 := \lambda\sum_{j=1}^{i} \phi^*_j(\lambda)f_j - a_i\sum_{j=1}^{i-1} \phi^*_{i-1,j}(\lambda)f_j + (a_i + b_i + d_i)\sum_{j=1}^{i} \phi^*_j(\lambda)f_j - b_i\sum_{j=1}^{i} \phi^*_{i+1,j}(\lambda)f_j.
$$

However, by using (2.23), $I_1$ reduces to

$$
I_1 = (\lambda v_i(\lambda) - a_i v_{i-1}(\lambda) + (a_i + b_i + d_i) v_i(\lambda) - b_i v_{i+1}(\lambda))
$$

$$
\times \sum_{j=1}^{i} u_j(\lambda)\pi_j f_j + a_i v_{i-1}(\lambda)u_i(\lambda)\pi_i f_i
$$

$$
= a_i v_{i-1}(\lambda)u_i(\lambda)\pi_i f_i.
$$
Similarly, $I_2$ reduces to

$$I_2 = -a_i u_{i-1}(\lambda) v_i(\lambda) \pi_i f_i.$$ 

Hence,

$$I_1 + I_2 = a_i \pi_i f_i u_{i-1}(\lambda) u_i(\lambda) \left[ \sum_{j=i+1}^{\infty} \frac{\rho_{j+1} - \rho_j}{u_j(\lambda) u_{j+1}(\lambda)} - \sum_{j=i}^{\infty} \frac{\rho_{j+1} - \rho_j}{u_j(\lambda) u_{j+1}(\lambda)} \right]$$

$$= a_i \pi_i (\rho_i - \rho_{i-1}) f_i = f_i,$$

which completes the proof of (A.18). We may prove (A.19) similarly.

**Proposition A.3.** Suppose that $f \in \mathcal{M}$ and $u_\infty(\lambda) = \lim_{n \to \infty} u_n(\lambda) < \infty$. Then

$$[\Phi^*(\lambda) f]_\infty = \lim_{n \to \infty} [\Phi^*(\lambda) f]_n = 0. \quad \text{(A.20)}$$

**Proof.** Since $u_\infty(\lambda) < \infty$ implies that $\rho_\infty < \infty$, by the definition of $v_n(\lambda)$ we obtain that

$$v_\infty(\lambda) := \lim_{n \to \infty} v_n(\lambda) = 0.$$ 

Hence, by (2.25), we get that

$$0 \leq (\lambda \Phi^*(\lambda) 1)_\infty = \lim_{n \to \infty} (\lambda \Phi^*(\lambda) 1)_n = 0,$$

which immediately yields (A.20).

We are now in a position to prove Lemma 2.4.

We first prove that the resolvent $\Phi^*(\lambda)$ constructed in (2.24) is indeed a $Q^*$-resolvent. The norm condition of $\Phi^*(\lambda)$ follows from (2.25) (which is proved in Proposition A.1) while the fact that $\Phi^*(\lambda)$ satisfies the Kolmogorov backward and forward equations associated with $Q^*$ follows directly from Proposition A.2. So we only need to verify that $\Phi^*(\lambda)$ satisfies the resolvent equation. To prove this, let $f \in \mathcal{M}$ and $F(\lambda) = \Phi^*(\lambda) f$. It is then easily seen that

$$F(\lambda) - F(\mu) + (\lambda - \mu) \Phi^*(\lambda) F(\mu) \in \mathcal{M},$$

which, by using (A.18), is a solution of the equation (2.10). This implies immediately by Lemma 2.1 that

$$F(\lambda) - F(\mu) + (\lambda - \mu) \Phi^*(\lambda) F(\mu) = c u(\lambda), \quad \text{(A.21)}$$

where $c$ is a scalar possibly depending on $\lambda$ and $\mu$ and $u = \{u_n(\lambda)\}$ is given in (2.11). If $u_\infty(\lambda) < \infty$, then letting $n \to \infty$ in (A.21) (i.e. letting the $n$th components of the vectors in (A.21) tend to infinity) and using Proposition A.3, we get $c u_\infty(\lambda) = 0$ and thus $c = 0$. If $u_\infty(\lambda) = +\infty$, then $c = 0$ follows from the fact that the left-hand side of (A.21) is bounded while $u(\lambda)$ is unbounded. Hence, we always have $c = 0$. Letting $f_i = \delta_{ij}$ yields the resolvent equation for $\Phi^*(\lambda)$. This proves that $\Phi^*(\lambda)$ is a $Q^*$-resolvent and satisfies both the Kolmogorov backward and forward equations associated with $Q^*$.

Secondly, we prove that $\Phi^*(\lambda)$ is the minimal $Q^*$-resolvent. Suppose that $\tilde{\Phi}(\lambda)$ is the minimal $Q^*$-resolvent. Since $\tilde{\Phi}(\lambda)$ and $\Phi^*(\lambda)$ both satisfy the Kolmogorov backward equation associated with $Q^*$, we have that, for fixed $j$, $\tilde{u}_i(\lambda) := \phi_{ij}^*(\lambda) - \tilde{\phi}_{ij}(\lambda)$ satisfies the following equation:

$$\lambda \tilde{u}_i(\lambda) - \sum_{k=1}^{\infty} q_{ik}^* \tilde{u}_k(\lambda) = 0.$$
Hence, \( \{ \overline{u}_i (\lambda) \} \) is a solution of (2.10) and, thus, by Lemma 2.1, we have \( \overline{u}_i (\lambda) = c(\lambda)u_i (\lambda) \), where \( u_i (\lambda) \) is given in (2.11). That is,

\[
\phi_{ij} (\lambda) = \phi_{ij} (\lambda) + c(\lambda)u_i (\lambda).
\]  

(A.22)

Now using the same trick as we did in connection with (A.21), we can also prove that \( c(\lambda) = 0 \) in (A.22). That is,

\[
\Phi^*(\lambda) = \tilde{\Phi}(\lambda).
\]

In other words, \( \Phi^*(\lambda) \) is indeed the minimal \( Q^* \)-resolvent.

Finally we prove the last part of Lemma 2.4, i.e. relations (2.25)-(2.29). First note that (2.25) is already proved in Proposition A.1 and (2.26) follows from (2.24), (2.20) and (2.12) directly. By virtue of (2.26), we immediately obtain (2.28) and (2.29) except that \( m_i < \infty \). However, the finiteness of these quantities can be easily shown. Indeed, as observed at the end of the proof of Theorem 5.1, we know that \( m_i \leq g_i \) (for \( i \geq 1 \)). Since \( R < \infty \) implies that all \( g_i, i \geq 1 \), are finite, so all \( m_i, i \geq 1 \), are finite, which then in turn implies (2.27). This completes the proof.

References


