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Existence, Uniqueness, and Constructions for Stochastically Monotone Q-Processes

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Abstract. The key problems in discussing duality and monotonicity for continuous-time Markov chains are to find conditions for existence and uniqueness and then to construct corresponding processes in terms of infinitesimal characteristics, i.e., q-matrices. Such problems are solved in this paper under the assumption that the given q-matrix is conservative. Some general properties of stochastically monotone Q-process (Q is not necessarily conservative) are also discussed.

Keywords: continuous-time Markov chains, monotone Q-processes, duality, Feller–Reuter–Riley functions, dual q-matrix, Feller q-matrix, conservative, existence, uniqueness, construction

1. Introduction

Duality is an important tool in Markov processes, in particular, in continuous-time Markov chains (CTMC) and interacting particle systems. Excellent references on this topic are [6] and [13]. It is revealed that duality has a close link with another important concept, i.e., monotonicity. It seems that Daley [8] was the first to emphasize the latter important concept.

The important fact concerning duality and monotonicity of CTMC is that, in most problems of interest, in particular, in applications, we only know the infinitesimal characteristic, i.e. the so-called q-matrix Q (for a formal definition, see below). Thus, when discussing duality and monotonicity, the crucial problem is the following: For a given q-matrix Q, under what condition does there exist a stochastically monotone Q-function, or does there exist a dual process? How can we obtain such processes if they do exist. More specifically, considering our dual problem only involves the totally stable q-matrix (see Sec. 2), we have to answer the following basic questions which have considerable significance in both theory and applications.

Question 1: What is the "necessary and sufficient" condition for a minimal Q-function to be stochastically monotone?

Question 2: If the minimal Q-function is not monotone, under what condition does there exist a (non-minimal) monotone Q-function?

Question 3: What is the condition for the monotone Q-function to be unique?

Question 4: How does one construct all the monotone Q-functions (for the given Q)?

Note that by Siegmund's result (see Theorem 1.1), the above questions can be posed equivalently by the "dual process".

It seems that Kirstein [12] was the first to give the answer to Question 1. Anderson [2] repeated Kirstein's result. However, for this relatively easy question, their results are not entirely correct, and thus, some amendments to their conclusions are necessary (see Remark 3.1 below). Anderson [2] also gives a partial solution to Question 3 (see Corollary 7.4.3 in [2]). Again, this partial solution is not entirely correct either, and thus, Question 3 remains open. As for Questions 2 and 4, to our knowledge, they have not yet been considered. Note that answering Question 4 properly is of particular importance since by obtaining the corresponding processes from the known condition we can then apply them to the application problems.

In this paper, we shall discuss the above four questions systematically. For reasons of simplicity, we assume that the given q-matrix Q is conservative, which is the most important case. We shall discuss the non-conservative case elsewhere. However, in Sec. 2, we shall not confine ourselves to the conservative case. Some general results shall be given there.

It is worth pointing out that stochastic monotonicity has important applications, particularly in birth and death processes. Excellent references on this topic are [18,19].

For simplicity, in this paper, we shall only consider CTMC on a linearly ordered state space. More specifically, we assume that the state space $E = \{0, 1, 2, \ldots\}$ with the natural ordering. The discussion of a more general case, where there only exists a partial ordering, is postponed to a subsequent paper. It should, however, be pointed out that our restriction to the linearly ordered case is not as limited as it looks. First, some very important applications concern only linearly ordered state space, for example, birth and death processes, queuing models, and branching processes. Second, there are some special properties for duality and monotonicity in the linearly ordered case which are not shared by general partial ordering state spaces. For example, only in the linearly ordering case are duality and monotonicity equivalent. These important special properties certainly justify the independent study of the linearly ordered case, perfecting its theory and application. Third, the benefit of concentrating on the linearly ordering case is undeniable since it makes our ideas and methods used in this paper in tackling duality and monotonicity more transparent and thus easier to generalize to the general partial ordering case. Finally, but more importantly, even for this linearly ordering case, as we mentioned above, many problems remain open and some commonly accepted conclusions concerning duality and monotonicity are not exactly correct and thus amendments are necessary.

We shall constantly use general conclusions in CTMC. An ideal reference of CTMC is [2]. We shall not hesitate to use general conclusions, notations, and terminologies in [2] without further explanations, for example, the usage of q-matrix and Q-matrix,

q-functions and Q-functions, q-resolvent and Q-resolvent, etc. (Note the difference between each pair!) Also, we speak of q-process (resp. Q-process) to denote either q-function (resp. Q-function) or q-resolvent (resp. Q-resolvent). This kind of usage was also seen in [14, 15]. Other excellent references on CTMC include [10, 11, 20, 21].

Section 2 contains some important preliminary results. Our main conclusions appear in Sec. 3. Examples are given in Sec. 4. We now give the basic definitions and notations used in this paper.

For CTMC on E, stochastical monotonicity can be defined simply as follows:

Definition 1.1. *A (standard) transition function $P(t) = \{p_{ij}(t);\ i, j\in E\}$ for a CTMC is said to be stochastically monotone if $\sum_{j=k}^{\infty} p_{ij}(t)$ is a non-decreasing function of i for each fixed $k\in E$ and $t \geq 0$.*

Note that honesty is not assumed for a transition function in Definition 1.1. In this paper "stochastically monotone" will be referred to as "monotone" when no confusion is caused. A fundamental result, discovered by Siegmund [17], is the following celebrated theorem.

Theorem 1.2. [17] *A transition function $P(t)$ is stochastically monotone if and only if there exists a dual transition function for $P(t)$, namely, if and only if there exists another (standard) transition function $\tilde{P}(t) = \{p_{ij}(t);\ i, j\in E\}$ such that*

$$\sum_{k=0}^{j} \tilde{p}_{ik}(t) = \sum_{k=i}^{\infty} p_{jk}(t) \quad (\forall i, j \in E, \quad \forall t \geq 0). \tag{1.1}$$

The importance of Siegmund's theorem lies in the fact that it reveals the close link between two q-processes. Indeed, by the proof of Siegmund's theorem, it is clear that a stochastically monotone q-function $P(t)$ and its dual $\tilde{P}(t)$ are totally determined by each other. Actually, by (1.1), we have

$$\tilde{p}_{ij}(t) = \sum_{k=i}^{\infty} (p_{jk}(t) - p_{j-1,k}(t)) \quad (\forall i, j \in E) \tag{1.2}$$

(define $p_{-1,k}(t) \equiv 0$) and that

$$p_{ji}(t) = \sum_{k=0}^{j} (\tilde{p}_{ik}(t) - \tilde{p}_{i+1,k}(t)) \quad (\forall i, j \in E). \tag{1.3}$$

The following relation is then easy to obtain:

$$p_{ji}(t) - p_{j-1,i}(t) = \tilde{p}_{ij}(t) - \tilde{p}_{i+1,j}(t). \tag{1.4}$$

It is worth pointing out that Siegmund's theorem can be stated in terms of its dual process.

Proposition 1.3. *Suppose $\tilde{P}(t)$ is a (standard) transition function which satisfies the following two conditions:*

(i) $\sum_{k=0}^{j} \tilde{p}_{ik}(t)$ *is a non-increasing function in i for each $j\in E$ and $t \geq 0$;* (1.5)

(ii) $\lim_{i\to\infty}\tilde{p}_{ij}(t) = 0 \quad (\forall j \in E, \ \ \forall t \geq 0).$ (1.6)

Then there exists a stochastically monotone transition function $P(t)$ such that (1.1)–(1.4) hold true.

Proposition 1.3 is just a "conjugation" for Siegmund's theorem. Its proof can be given in exactly the same way as for Theorem 1.2 and thus is omitted here.

A q-function satisfying (1.6) is usually called a Feller–Reuter–Riley function (see Sec. 2).

Following [2], we introduce the following notations:

$D = \{$all stochastically monotone transition functions$\}$ (1.7)

$\tilde{D} = \{$all Feller–Reuter–Riley transition function $P(t)$ such that

$\sum_{k=0}^{j} p_{ik}(t)$ is non-increasing in i for each j and $t\}$. (1.8)

Then (1.2) defines a one-to-one mapping of D onto $\tilde{D}$ with inverse given by (1.3).

The elements of D and $\tilde{D}$ are called duals of one another.

On the other hand, it is well known that, for any standard transition function $P(t)$, the limit

$$\lim_{t\to 0}(P(t) - I)/t \tag{1.9}$$

exists and this limit matrix, $Q = \{q_{ij};\, i, j{\in}E\}$, usually called a q-matrix, satisfies the following conditions:

$$0 \leq q_{ij} < +\infty \quad (i{\neq}j) \tag{1.10}$$

$$\sum_{j\neq i} q_{ij} \leq -q_{ii} \leq +\infty \quad (\forall i \in E). \tag{1.11}$$

Let $q_i = -q_{ii} \quad (i{\in}E)$. Note that, presently, $q_i = +\infty$ is possible for some i (or even for all $i{\in}E$). However, we shall immediately see that such a case will not occur when discussing stochastic monotonicity (duality) (see Sec. 2). When all $q_i \ \ (i{\in}E)$ are finite, the q-matrix Q is called *totally stable* and, furthermore, if

$$\sum_{j\neq i} q_{ij} = -q_i < +\infty \quad (\forall i \in E), \tag{1.12}$$

then Q is called conservative. It is also well known that, for a totally stable q-matrix Q, there always exists a standard transition function $P(t)$, called *Feller minimal Q-function*, such that (1.9) holds true.

It is convenient to introduce the following notations in line with (1.7) and (1.8):

$D(Q) = \{$all the stochastically monotone transition functions with given q-matrix $Q\}$. (1.13)

So Questions 1 and 2 involve to investigate the conditions under which $D(Q){\neq}\varnothing$ (the empty set) for a given Q, while Question 3 is concerned with finding conditions under which $|D(Q)| = 1$, where $|.|$ denotes the cardinal number of a set.

Suppose that $D(Q){\neq}\varnothing$ for a given Q and let $P(t){\in}D(Q)$. Then, by Theorem 1.2, there exists another transition function $\tilde{p}(t) \in \tilde{D}$ such that (1.1)–(1.4) hold true. This $\tilde{P}(t)$ has its own q-matrix, $\tilde{Q}$ say. We shall use the following term to compile all the above information:

$$P(t) \in D(Q) \text{ and } \tilde{P}(t) \in \tilde{D}(P(t); Q; \tilde{Q}). \tag{1.14}$$

The meaning of (1.14) should be clear and unambiguous.

2. General Results

Before formally investigating the existence, uniqueness, and constructions of monotone Q-functions, we first consider some general properties for monotonicity. Note that we have not made any assumptions about q-matrix Q, even some $q_i = +\infty$ might be possible at the present stage. Our first aim is to exclude the so-called non-totally stable case. Indeed, it can be easily proved (see Theorem 2.1 below) that such a case will not occur when discussing monotonicity.

Suppose $P(t)$ is stochastically monotone. Then by Siegmund's theorem, there exists another $\tilde{P}(t)$ such that (1.1) holds true. Let the q-matrices of $P(t)$ and $\tilde{P}(t)$ be Q and $\tilde{Q}$, respectively. Then we have the following simple relations:

$$q_{ji} - q_{j-1,i} = \tilde{q}_{ij} - \tilde{q}_{i+1,j} \quad (\forall i, j \in E) \tag{2.1}$$

(with $q_{-1,j} \equiv 0 \ \forall j$)

$$q_{ij} = \sum_{k=0}^{i} (\tilde{q}_{jk} - \tilde{q}_{j+1,k}) \quad (\forall i, j \in E) \tag{2.2}$$

which are easily obtained by (1.4) and (1.3), respectively.

Note that, although Q is determined by $\tilde{Q}$ through (2.2), $\tilde{Q}$ generally cannot be simply determined by Q although we have (1.2). What we can obtain from (1.2) is the following inequality (see Lemma 2.4):

$$\tilde{q}_{ij} \geq \sum_{k=i}^{\infty} (q_{jk} - q_{j-1,k}) \quad (\forall i, j \in E). \tag{2.3}$$

In fact, the conditions under which (2.3) become an equality is a crucial problem in our discussion. Now, the first important conclusion, due to [1], is

Theorem 2.1. [1] *If $P(t)$ is stochastically monotone, then its dual $\tilde{P}(t)$ is a Feller–Reuter–Riley q-function. Hence, $\tilde{Q}$ must be totally stable and $\tilde{P}(t)$ is the Feller–Reuter–Riley minimal $\tilde{Q}$-function.*

Proof. See [2]. □

Remark 2.2. To our knowledge, Anderson [1] was the first to discover the interesting link between duality and Feller–Reuter–Riley q-functions. Here, we use the term "Feller–Reuter–Riley function" since it was first systematically discussed in the important paper of Reuter and Riley [16]. This paper has far-reaching significance (see [3] for further details and other interesting applications).

Since $\tilde{Q}$ is totally stable, then by (2.2), we immediately have that Q itself is also totally stable. Hence, in discussing monotonicity, we only need to consider the totally stable case. Thus, we are fully justified in discussing Kolmogorov backward and/or forward equations, Feller minimal q-functions, etc.

An important fact we observe is the following interesting result:

Theorem 2.3. *If $P(t)$ is stochastically monotone, then $P(t)$ must satisfy Kolmogorov forward equations.*

Proof. We claim that, if $P(t) \in D(Q)$, then

$$\frac{d}{dt} p_{ij}(t) = \sum_{k=0}^{\infty} p_{ik}(t) q_{kj} \quad (\forall i, j \in E, \ \ \forall t \geq 0). \tag{2.4}$$

Indeed, let $\tilde{P}(t) \in \tilde{D}(P(t); Q; \tilde{Q})$. Then since $\tilde{P}(t)$ is the Feller minimal $\tilde{Q}$-function and thus satisfies Kolmogorov's backward equations, i.e.,

$$\frac{d}{dt} \tilde{p}_{ij}(t) = \sum_{k=0}^{\infty} \tilde{q}_{ik} \tilde{p}_{kj}(t) \quad (\forall i, j \in E, \ \ \forall t \geq 0). \tag{2.5}$$

We then have, by (1.3),

$$\begin{aligned}
\frac{d}{dt} p_{ij}(t) &= \frac{d}{dt} \sum_{k=0}^{i} \big(\tilde{p}_{jk}(t) - \tilde{p}_{j+1,k}(t)\big) = \sum_{k=0}^{i} \Big[\frac{d}{dt} \tilde{p}_{jk}(t) - \frac{d}{dt} \tilde{p}_{j+1,k}(t)\Big] \\
&= \sum_{k=0}^{i} \left[\sum_{l=0}^{\infty} \tilde{q}_{jl} \tilde{p}_{lk}(t) - \sum_{l=0}^{\infty} \tilde{q}_{j+1,l} \tilde{p}_{lk}(t) \right] \quad \text{(by (2.5))} \\
&= \sum_{k=0}^{i} \left[\sum_{l=0}^{\infty} \big(\tilde{q}_{jl} - \tilde{q}_{j+1,l}\big) \tilde{p}_{lk}(t) \right] = \sum_{l=0}^{\infty} \big(\tilde{q}_{jl} - \tilde{q}_{j+1,l}\big) \cdot \sum_{k=0}^{i} \tilde{p}_{lk}(t) \\
&= \sum_{l=0}^{\infty} \big(\tilde{q}_{jl} - \tilde{q}_{j+1,l}\big) \cdot \sum_{k=l}^{\infty} p_{ik}(t) \quad \text{(see (1.1))} \\
&= \sum_{k=0}^{\infty} \left[\sum_{l=0}^{k} p_{ik}(t) \big(\tilde{q}_{jl} - \tilde{q}_{j+1,l}\big) \right] = \sum_{k=0}^{\infty} p_{ik}(t) \cdot \sum_{l=0}^{k} \big(\tilde{q}_{jl} - \tilde{q}_{j+1,l}\big) \\
&= \sum_{k=0}^{\infty} p_{ik}(t) \sum_{l=0}^{k} \big(q_{lj} - q_{l-1,j}\big) \quad \text{(see (2.1))} \\
&= \sum_{k=0}^{\infty} p_{ik}(t) q_{kj} \quad (\text{recall } \ q_{-1,j} \equiv 0),
\end{aligned}$$

which proves (2.4). □

Note. In the above and elsewhere in this paper, we have constantly used associated and distributive law which are not true in all cases. The justification for these can be easily checked. Such trivial detailed verification will be omitted here and elsewhere.

We now turn to Kolmogorov backward equations. First, we point out a relation between Q and $\tilde{Q}$.

Lemma 2.4. *Let $P(t)$ be a monotone q-function and $\tilde{P}(t)$ its dual. Further, let Q and $\tilde{Q}$ be the q-matrices of $P(t)$ and $\tilde{P}(t)$, respectively. Then the limit $\lim_{n\to\infty} \tilde{q}_{nj}$ denoted by $\tilde{C}_j$ exists and*

$$0 \leq \tilde{C}_j < +\infty \quad (\forall j \in E). \tag{2.6}$$

Furthermore,

$$\tilde{q}_{ij} = \tilde{C}_j + \sum_{k=i}^{\infty}(q_{jk} - q_{j-1,k}) \quad (\forall i, j \in E). \tag{2.7}$$

Proof. By (2.1), for any n,

$$\sum_{k=i}^{n}(q_{jk} - q_{j-1,k}) = \sum_{k=i}^{n}(\tilde{q}_{kj} - \tilde{q}_{k+1,j})$$

or

$$\sum_{k=i}^{n}(q_{jk} - q_{j-1,k}) = \tilde{q}_{ij} - \tilde{q}_{n+1,j} \tag{2.8}$$

Letting $n \to \infty$ in (2.8) yields the fact that the left-hand side of (2.8) has a finite limit, and so does the right-hand side of (2.8). But $\tilde{q}_{ij}$ is fixed when $n \to \infty$ in (2.8), thus the limit $\lim_{n\to\infty}\tilde{q}_{nj}$ exists and the limit is finite. The non-negative property of this limit is obvious. Assertion (2.6) is thus obtained and (2.7) is just the limiting form of (2.8). □

Comparing (2.7) with (2.3) shows that we have proved (2.3) and that we actually give more information about $\tilde{Q}$ in (2.7). In particular, if $\tilde{C}_j = 0(\forall j)$, we obtain

$$\tilde{q}_{ij} = \sum_{k=i}^{\infty}(q_{jk} - q_{j-1,k}) \quad (\forall i, j \in E). \tag{2.9}$$

Now, an important question arises: Under what conditions will (2.9) hold true? Recall the definition [2,16] that a totally stable q-matrix Q is called a Feller q-matrix if

$$\lim_{n\to\infty} q_{nj} = 0 \quad (\forall i, j \in E).$$

So, if $\tilde{Q}$ is Feller, we obtain the simple form (2.9).

Interestingly, we may claim the following conclusion which answers the question when (2.9) holds true.

Theorem 2.5. *Let $P(t)$ be a monotone q-function and $\tilde{P}(t)$ its dual. Further, let Q and $\tilde{Q}$ be the q-matrices of $P(t)$ and $\tilde{P}(t)$, respectively. Then $P(t)$ satisfies Kolmogorov backward equations if and only if $\tilde{Q}$ is a Feller matrix. In such a case, we have*

$$\tilde{q}_{ij} = \sum_{k=i}^{\infty}(q_{jk} - q_{j-1,k}) \quad (\forall i, j \in E). \tag{2.10}$$

Proof. "If" part. If $\tilde{Q}$ is Feller, then (2.10) holds true. By using (1.3) again and recalling that $\tilde{P}(t)$ is the Feller minimal $\tilde{Q}$-function and thus satisfies Kolmogorov forward equations,

$$\begin{aligned}
\frac{d}{dt}p_{ij}(t) &= \sum_{k=0}^{i}\left[\frac{d}{dt}\tilde{p}_{jk}(t) - \frac{d}{dt}\tilde{p}_{j+1,k}(t)\right] = \sum_{k=0}^{i}\left[\sum_{l=0}^{\infty}\tilde{p}_{jl}(t)\tilde{q}_{lk} - \sum_{l=0}^{\infty}\tilde{p}_{j+1,l}(t)\tilde{q}_{lk}\right] \\
&= \sum_{l=0}^{\infty}\left(\tilde{p}_{jl}(t) - \tilde{p}_{j+1,l}(t)\right)\cdot\sum_{k=0}^{i}\tilde{q}_{lk} \\
&= \sum_{l=0}^{\infty}\left(\tilde{p}_{jl}(t) - \tilde{p}_{j+1,l}(t)\right)\cdot\sum_{k=0}^{i}\sum_{m=l}^{\infty}(q_{km} - q_{k-1,m}) \quad \text{(by (2.10))} \\
&= \sum_{l=0}^{\infty}\left(p_{lj}(t) - p_{l-1,j}(t)\right)\cdot\sum_{m=l}^{\infty}q_{im} \quad \text{(by (1.4))} \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\left(p_{lj}(t) - p_{l-1,j}(t)\right)\cdot q_{im} = \sum_{m=0}^{\infty}q_{im}\cdot p_{mj}(t).
\end{aligned}$$

"Only if" part. Let $P(t)\in D(Q)$ and $\tilde{P}(t)\in\tilde{D}(P(t);Q;\tilde{Q})$. Then, by (1.2),

$$\tilde{p}_{ij}(t) = \sum_{k=i}^{\infty}(p_{jk}(t) - p_{j-1,k}(t)) \quad (\forall i, j \in E). \tag{2.11}$$

By introducing the deficit function

$$\sigma_j(t) = 1 - \sum_{k=0}^{\infty}p_{jk}(t) \quad (\forall i, j \in E, \ \ \forall t \geq 0) \tag{2.12}$$

$$(\sigma_{-1}(t) \equiv 1),$$

Eq. (2.11) can be written as

$$\tilde{p}_{ij}(t) = \left(1 - \sigma_j(t) - \sum_{k=0}^{i-1}p_{jk}(t)\right) - \left(1 - \sigma_{j-1}(t) - \sum_{k=0}^{i-1}p_{j-1,k}(t)\right). \tag{2.13}$$

Now, if $i > j$, by dividing t on both sides of (2.13), we obtain

$$\begin{aligned}
\frac{\tilde{p}_{ij}(t)}{t} &= \frac{1 - p_{jj}(t)}{t} - \frac{\sigma_j(t)}{t} - \sum_{k=0(k\neq j)}^{i-1}\frac{p_{jk}(t)}{t} \\
&\quad - \left(\frac{1 - p_{j-1,j-1}(t)}{t} - \frac{\sigma_{j-1}(t)}{t} - \sum_{k=0(k\neq j-1)}^{i-1}p_{j-1,k}(t)\right)
\end{aligned}$$

Letting $t\to 0$ then yields

$$\tilde{q}_{ij} = \left(-\sigma_j - \sum_{k=0}^{i-1}q_{jk}\right) - \left(-\sigma_{j-1} - \sum_{k=0}^{i-1}q_{j-1,k}\right), \tag{2.14}$$

where

$$\sigma_j = \frac{d}{dt}\sigma_j(t)\Big|_{t=0} \tag{2.15}$$

If $i < j$, then rewrite (2.13) as

$$\tilde{p}_{ij}(t) = \left(-\sigma_j(t) - \sum_{k=0}^{i-1} p_{jk}(t)\right) + \left(\sigma_{j-1}(t) + \sum_{k=0}^{i-1} p_{j-1,k}(t)\right),$$

and the same procedure again yields (2.14).

If $i = j$, just rewrite (2.13) as

$$\tilde{p}_{ii}(t) - 1 = \left(-\sigma_j(t) - \sum_{k=0}^{i-1} p_{jk}(t)\right) - \left(1 - p_{j-1,j-1}(t) - \sigma_{j-1}(t) - \sum_{k=0(k\neq j-1)}^{i-1} p_{j-1,k}(t)\right).$$

Then, dividing by t and letting $t\to 0$, we obtain the same relation as (2.14). Hence, (2.14) holds true for all cases. It is well known [21] that a Q-function $P(t)$ satisfies Kolmogorov's backward equations if and only if

$$\sigma_j = d_j \quad (\forall j \in E), \tag{2.16}$$

where

$$d_i = q_i - \sum_{j\neq i} q_{ij} = -\sum_{j=0}^{\infty} q_{ij}. \tag{2.17}$$

So (2.14) can be written as

$$\tilde{q}_{ij} = \sum_{k=i}^{\infty} q_{jk} - \sum_{k=i}^{\infty} q_{j-1,k}$$

or

$$\tilde{q}_{ij} = \sum_{k=i}^{\infty} (q_{jk} - q_{j-1,k}). \tag{2.18}$$

Comparing with (2.7) then yields

$$\tilde{C}_j \equiv 0 \quad (\forall j \in E),$$

i.e., $\tilde{Q}$ is Feller. □

We then obtain the following important corollary:

Corollary 2.6. *Suppose $P(t)$ is a monotone q-function with q-matrix $Q = \{q_{ij}; i, j \in E\}$ and that it satisfies Kolmogorov backward equations. Then we have*

$$\sum_{j=k}^{\infty} q_{ij} \le \sum_{j=k}^{\infty} q_{i+1,j} \quad (\forall i, k \in E \text{ such that } k \neq i+1). \tag{2.19}$$

Furthermore, (2.19) is equivalent to

$$\sum_{j=k}^{\infty} q_{ij} \le \sum_{j=k}^{\infty} q_{mj} \tag{2.20}$$

whenever $i \le m$, and k is such that either $k \le i$ or $k > m$.

Proof. Noting (2.10) and the fact that, for $i \neq j$, $\tilde{q}_{ij} \ge 0$, we immediately obtain (2.19). It is also easy to see that (2.19) and (2.20) are equivalent. □

Now, recall the following definition in [2].

Definition 2.7. *A totally stable q-matrix Q (not necessarily conservative) is called monotone if (2.19) or, equivalently, (2.20) holds true.*

Before proceeding further, we point out another equivalent condition for monotonicity via q-resolvents rather than q-functions which is more convenient to use in some cases.

Proposition 2.8. *The following two statements are equivalent:*

(i) *The transition function $P(t) = \{p_{ij}(t); i, j \ge 0\}$ is monotone, i.e., $\sum_{j=k}^{\infty} p_{ij}(t)$ is a non-decreasing function of i for each fixed $k \in E$ and $t \ge 0$.*

(ii) *The q-resolvent $R(\lambda) = \{r_{ij}(\lambda); i, j \ge 0\}$ of $P(t)$ satisfies the condition that $\sum_{j=k}^{\infty} \lambda\, r_{ij}(\lambda)$ is a non-decreasing function of i for each fixed $k \in E$ and $\lambda > 0$, where the resolvent $R(\lambda)$, by definition, is given by*

$$r_{ij}(\lambda) = \int_0^{\infty} e^{-\lambda t} p_{ij}(t)dt \quad (\lambda > 0). \tag{2.21}$$

Proof. (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (i) Use the dual technique. □

The details of the proof are omitted since the proof is similar to Siegmund's theorem, except one uses q-resolvent instead of transition functions.

Parallel to Proposition 2.8, we have the following conclusion:

Proposition 2.9. *The following two statements are equivalent:*

(i) *The transition function $P(t) = \{p_{ij}(t); i, j \ge 0\}$ is stochastically decreasing, i.e., $\sum_{j=0}^{k} p_{ij}(t)$ is a non-increasing function of i for each fixed $k \in E$ and $t \ge 0$.*

(ii) *The q-resolvent $R(\lambda) = \{r_{ij}(\lambda); i, j \ge 0\}$ of $P(t)$ satisfies the condition that $\sum_{j=0}^{k} \lambda r_{ij}(\lambda)$ is a non-increasing function of i for each fixed $k \in E$ and $\lambda > 0$.*

Proof. The proof is similar to Proposition 2.8 and is thus omitted. □

3. Existence, Uniqueness and Construction

We now formally discuss the existence, uniqueness, and construction of monotone transition functions when q-matrix Q is given. The final conclusion is Theorem 3.15 which answers all four basic questions posted in Sec. 1. We repeat that, by Theorem 2.1, we need only consider totally stable case.

Throughout this section, we assume that Q is conservative since this is the most important case, in particular, from the viewpoint of applications.

We first discuss the uniqueness problem. Note that by Siegmund's theorem, there exists a one-to-one correspondence between a monotone transition function and its dual. Furthermore, this dual is the minimal $\tilde{Q}$-function, which is unique. However, this does not immediately imply uniqueness for monotone Q-functions. Indeed, there might exist two or more monotone q-functions $P_1(t)$ and $P_2(t)$ with the same q-matrix Q but their dual, $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$ say, have different q-matrices $\tilde{Q}_1$ and $\tilde{Q}_2$.

Fortunately, this will not occur when Q is conservative due to Theorem 2.5.

Theorem 3.1. *For a given conservative q-matrix Q, if there exists a monotone Q-function, then it must be unique.*

Proof. It is well known that, for a conservative q-matrix Q, any Q-function satisfies Kolmogorov backward equations. Now, suppose there exist two monotone Q-functions $P_1(t)$ and $P_2(t)$. Let their duals be $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$ with q-matrices $\tilde{Q}_1 = \{\tilde{q}_{ij}^{(1)}\}$ and $\tilde{Q}_2 = \{\tilde{q}_{ij}^{(2)}\}$, respectively. Since both $P_1(t)$ and $P_2(t)$ satisfy Kolmogorov backward equations, by Theorem 2.5, we know that (see (2.10))

$$\tilde{q}_{ij}^{(1)} = \sum_{k=i}^{\infty}(q_{jk} - q_{j-1,k}) = \tilde{q}_{ij}^{(2)} \quad (\forall i, j \in E),$$

i.e.,

$$\tilde{Q}_1 = \tilde{Q}_2.$$

Thus, both $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$ are Feller minimal $\tilde{Q}_1(= \tilde{Q}_2)$-functions. Therefore,

$$\tilde{P}_1(t) \equiv \tilde{P}_2(t).$$

Now, by Siegmund's theorem, we obtain $P_1(t) \equiv P_2(t)$, since the correspondence between a monotone transition function and its dual is one-to-one. □

Theorem 3.1 answers Question 3 in Sec. 1.

The existence problem is more difficult. Let us first consider Feller minimal transition functions. The following theorem answers Question 1.

Theorem 3.2. *For a given conservative q-matrix Q, the Feller minimal Q-function is stochastically monotone if and only if the following two conditions hold true:*

(i) *Q is regular;*
(ii) *Q is monotone.*

(Recall Definition 2.7 for the meaning of a q-matrix Q being monotone.)

Remark 3.3. Condition (ii) was first given by Kirstein [12] (see also [2]). As for condition (i), it seems that Kirstein [12] did not realize that condition (i) is actually necessary for the minimal Q-function to be monotone. Indeed, he made condition (i) as an assumption in [12]. In trying to polish Kirstein's result, Anderson [2] seemed to deem condition (i) as superfluous. Unfortunately, without condition (i), the theorem fails to be true. A counterexample can be easily given (see Example 4.1.).

Remark 3.4. Conditions for a conservative q-matrix Q being regular (condition (i)) are well known. Indeed, it is equivalent to say that the Feller minimal Q-function is honest and thus unique. Another equivalent condition which is commonly used is that the equation

$$\begin{cases} (\lambda I - Q)U = \mathbf{0} \\ \mathbf{0} \le U \le \mathbf{1} \end{cases} \tag{3.2}$$

has no non-trivial solution for some (and therefore, for all) $\lambda > 0$. (For details and other equivalent conditions, see, for example, Theorem 2.2.7 in [2].)

In order to illustrate the necessity of condition (i), we first give a somewhat more general result before proving Theorem 3.2.

Proposition 3.5. *Suppose Q is a conservative q-matrix and $P(t)$ is a stochastically monotone Q-function (not necessarily minimal), then $P(t)$ must be honest, i.e.,*

$$\sum_{j=0}^{\infty} p_{ij}(t) = 1 \quad (\forall i \in E,\ \ t \ge 0). \tag{3.3}$$

Proof. Let $P(t) \in D(Q)$ and $\tilde{P}(t) \in \tilde{D}(P(t); Q; \tilde{Q})$. Since Q is conservative, we know that $P(t)$ must satisfy Kolmogorov backward equations. Hence, by Theorem 2.5 (see (2.10)),

$$\tilde{q}_{00} = \sum_{k=0}^{\infty} q_{0k} = 0.$$

Thus, 0 is an absorbing state of $\tilde{P}(t)$ and hence,

$$\tilde{p}_{00}(t) \equiv 1 \quad \tilde{p}_{0j}(t) \equiv 0 \quad (\forall j \in E,\ \ \forall t \ge 0).$$

By noting (1.1), this immediately leads to

$$\sum_{k=0}^{\infty} p_{jk}(t) = \sum_{k=0}^{j} \tilde{p}_{0k}(t) = \tilde{p}_{00}(t) = 1 \quad (\forall j \in E,\ \ \forall t \ge 0).$$

The conclusion follows. □

Now, we give the proof of Theorem 3.2.

Proof. Necessity. Condition (i) follows from Proposition 3.5 and condition (ii) follows from Corollary 2.6.

Sufficiency. Omitted. □

Remark 3.6. The reason for omitting the sufficiency of Theorem 3.2 is because the proof can be easily obtained by amending Anderson's original proof. In fact, Anderson's proof is correct except the last step. The mistake made by Anderson was that the monotone convergence theorem was incorrectly used in the last step. This difficulty can be easily overcome by simply reversing the direction of the inequality. Of course, here, condition (i) guarantees that the procedure will be successful. This is precisely where condition (i) works. Another reason for omitting the proof is because the proof of sufficiency can be covered by our approach in proving Theorem 3.7 below, subject to some obvious amendments. Anderson's approach can only be applied to the Feller minimal Q-function case since he uses the asymptotic technique. However, our new approach, by using the dual technique, can cover all cases.

We now turn to the more awkward problem: existence of non-minimal monotone Q-functions. To our knowledge, this problem has not been considered in the literature and thus remains open. The following theorem answers Question 2.

Theorem 3.7. *For a given conservative q-matrix Q, there exists a non-minimal stochastically monotone Q-function if and only if the following three conditions are satisfied:*

(ii) *Q is monotone;*

(ii) *the equation*

$$\begin{cases} (\lambda I - Q)U = \mathbf{0} \\ \mathbf{0} \le U \le \mathbf{1} \end{cases} \tag{3.4}$$

has a non-trivial solution for some (and therefore, for all) $\lambda > 0$;

(iii) *the equation*

$$\begin{cases} V(\lambda I - Q) = \mathbf{0} \\ \mathbf{0} \le V \in l \end{cases} \tag{3.5}$$

has a non-trivial solution for some (and therefore, for all) $\lambda > 0$.

Remark 3.8. For definitions and applications of Eqs. (3.4) and (3.5), see [10] or [21]. Of course, for a particular Q, whether Eqs. (3.4) and/or (3.5) have a non-trivial solution requires further consideration. For the birth-death or branching processes, we have convenient conditions to check them (see Sec. 4).

The proof of Theorem 3.7 is postponed until after Lemmas 3.9 to 3.12.

Lemma 3.9 *Let $Q = \{q_{ij}; i, j \in E\}$ be a conservative monotone q-matrix (recall Definition 2.7) and define another matrix $\tilde{Q} = \{\tilde{q}_{ij}; i, j \in E\}$ by*

$$\tilde{q}_{ij} = \sum_{k=i}^{\infty} (q_{jk} - q_{j-1,k}) \quad (\forall i, j \in E), \tag{3.6}$$

where $q_{-1,k} \triangleq 0$. Then

(i) *$\tilde{Q}$ is a totally stable Feller q-matrix;* (3.7)

(ii) *$\tilde{q}_{0j} \equiv 0 \quad (\forall j \in E)$;* (3.8)

(iii) *$\sum_{k=0}^{j} \tilde{q}_{ik} = \sum_{m=i}^{\infty} q_{jm}$;* (3.9)

(iv) $\sum_{k=0}^{j} \tilde{q}_{ik} \geq \sum_{k=0}^{j} \tilde{q}_{i+1,k} \quad (\forall j \neq i)$; (3.10)

(v) $\sum_{k=0}^{j} \tilde{q}_{ik} \leq \sum_{k=0}^{j} \tilde{q}_{i+1,k} \quad (\forall j = i)$; (3.11)

and thus, if Q is Feller, then $\tilde{Q}$ is conservative.

Proof. (i) (3.7) is easy and can also be seen in [2].

(ii) (3.8) is derived from (3.6) and the fact that Q is conservative.

(iii) Summing (3.6) immediately yields (3.9). Considering Q is conservative, (3.9) can be rewritten as

$$\sum_{k=0}^{j} \tilde{q}_{ik} = -\sum_{m=0}^{i-1} q_{jm} \quad (\forall i \geq 1). \tag{3.12}$$

Now, if Q is Feller, then $q_{jm} \to 0$ $(\forall m)$ when $j \to \infty$, and hence, by letting $j \to \infty$ in (3.12), we obtain

$$\sum_{k=0}^{\infty} \tilde{q}_{ik} = 0 \quad (\forall i \geq 1). \tag{3.13}$$

This, together with (3.8), shows that $\tilde{Q}$ is conservative.

(iv) By (3.9), we have

$$\sum_{k=0}^{j} (\tilde{q}_{ik} - \tilde{q}_{i+1,k}) = \sum_{m=i}^{\infty} q_{jm} - \sum_{m=i+1}^{\infty} q_{jm} = q_{ji}.$$

If $j \neq i$, then $q_{ji} \geq 0$ which yields (3.10), while, if $j = i$, then $q_{ji} \leq 0$ which yields (3.11). □

In the sequel, when Q is conservative and monotone, we call $\tilde{Q}$, defined in (3.6), the dual q-matrix of Q. The main point here is that $\tilde{Q}$ is also a totally stable q-matrix, and thus, we may discuss the minimal $\tilde{Q}$-function, etc. Of course, at present, the dual q-matrix is just a definition as in (3.6), it is not involved with dual processes. However, we shall immediately see the importance of dual q-matrix $\tilde{Q}$ in considering monotone Q-functions.

The following two interesting lemmas play an important role in proving Theorems 3.7. They show that there exists a somewhat "symmetric" behavior between the entrance Martin boundary and exit Martin boundary of Q and its dual q-matrix $\tilde{Q}$ — a phenomenon observed by Cox and Röster [7].

Lemma 3.10. *Let Q be a conservative monotone q-matrix and $\tilde{Q}$ its dual q-matrix defined as in (3.6). Suppose the equation*

$$\begin{cases} \tilde{V}(\lambda I - \tilde{Q}) = \mathbf{0} \\ \mathbf{0} \leq \tilde{V} \in l \end{cases} \tag{3.14}$$

has a non-trivial solution for some (and therefore, for all) $\lambda > 0$, then the equation

$$\begin{cases} (\lambda I - Q)U = \mathbf{0} \\ \mathbf{0} \leq U \leq \mathbf{1} \end{cases} \tag{3.15}$$

also has a non-trivial solution for some (and therefore, for all) $\lambda > 0$. Moreover, we have

$$\lim_{i\to\infty} \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = 0, \tag{3.16}$$

and thus, the minimal Q-function $F(t)$ is a Feller–Reuter–Riley function, where $\Phi(\lambda) = \{\Phi_{ij}(\lambda)\}$ is the minimal Q-resolvent.

Proof. Suppose, for a fixed $\lambda > 0$, Eq. (3.14) has a non-trivial solution, $\tilde{V}(\lambda) = \{\tilde{v}_j(\lambda);\ j \geq 0\}$ say. Then

$$0 \leq \tilde{v}_j(\lambda) < +\infty, \quad \tilde{v}_j(\lambda) \not\equiv 0, \text{ and } \sum_{j=0}^{\infty} \tilde{v}_j(\lambda) \triangleq C(\lambda) < +\infty.$$

Define $\bar{U}(\lambda) = \{u_j(\lambda);\ j \geq 0\}$ as follows:

$$u_j(\lambda) = \frac{1}{C(\lambda)} \sum_{k=0}^{j} \tilde{v}_k(\lambda) \quad (\forall j \geq 0), \tag{3.17}$$

then it is easy to see that

$$u_j(\lambda) \geq 0 \quad (\forall j \geq 0), \ \ u_j(\lambda) \not\equiv 0, \ \text{ and } \ u_j(\lambda) \uparrow 1 \ (j \to \infty),$$

i.e.,

$$0 \leq \bar{U}(\lambda) \leq 1 \ \text{ and } \ \bar{U}(\lambda) \not\equiv \mathbf{0}. \tag{3.18}$$

Now, we claim that the $\bar{U}(\lambda)$ defined in (3.17) satisfies

$$(\lambda I - Q)\bar{U} = 0 \tag{3.19}$$

Indeed, since $\tilde{V}(\lambda)$ satisfies (3.14), i.e.,

$$\lambda \tilde{v}_k(\lambda) = \sum_{i=0}^{\infty} \tilde{v}_i(\lambda)\tilde{q}_{ik} \quad (\forall k \geq 0).$$

therefore,

$$\begin{aligned}
\lambda \sum_{k=0}^{j} \tilde{v}_k(\lambda) &= \sum_{k=0}^{j} \sum_{i=0}^{\infty} \tilde{v}_i(\lambda)\tilde{q}_{ik} = \sum_{i=0}^{\infty} \tilde{v}_i(\lambda) \cdot \sum_{k=0}^{j} \tilde{q}_{ik} \\
&= \sum_{i=0}^{\infty} \tilde{v}_i(\lambda) \cdot \sum_{m=i}^{\infty} q_{jm} \quad \text{(by (3.9))} \\
&= \sum_{m=0}^{\infty} q_{jm} \cdot \sum_{i=0}^{m} \tilde{v}_i(\lambda).
\end{aligned}$$

Noting (3.17), we see that

$$\lambda C(\lambda) u_j(\lambda) = \sum_{m=0}^{\infty} q_{jm} C(\lambda) u_m(\lambda)$$

or

$$\lambda u_j(\lambda) = \sum_{m=0}^{\infty} q_{jm} u_m(\lambda) \quad (\text{since } C(\lambda) > 0).$$

Hence, (3.19) holds true. Now, (3.19), together with (3.18), shows that, for this fixed (and therefore, for all) $\lambda > 0$, Eq. (3.15) has a non-trivial solution. Moreover, it is well known that (3.15) has a maximal solution, $\bar{X}(\lambda) = \{\bar{X}_j(\lambda);\ j \geq 0\}$ say. By the maximal property, we certainly have

$$u_i(\lambda) \leq \bar{X}_i(\lambda) \leq 1 \quad (\forall i \geq 0).$$

Since $\lim_{i\to\infty} u_i(\lambda) = 1$, we obtain $\lim_{i\to\infty} \bar{X}_i(\lambda) = 1$. But Q is conservative and so [21]

$$1 - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = \bar{X}_i(\lambda)$$

and thus $\lim_{i\to\infty} \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = 0.$ (3.20)

By (3.20), we have (actually much stronger than)

$$\lim_{i\to\infty} \phi_{ij}(\lambda) = 0 \quad (\forall j \in E).$$

Thus, by [16], the minimal Q-function is a Feller–Reuter–Riley function. □

Similarly, we have

Lemma 3.11. *Let $Q = \{q_{ij}\}$ be a conservative monotone q-matrix, and $\tilde{Q}$ its dual q-matrix defined as in (3.6). Suppose the equation*

$$\begin{cases} V(\lambda I - Q) = \mathbf{0} \\ \mathbf{0} \leq V \in l \end{cases} \tag{3.21}$$

has a non-trivial solution for some (and therefore, for all) $\lambda > 0$, then the equation

$$\begin{cases} (\lambda I - \tilde{Q})U = \tilde{D} \\ \mathbf{0} \leq U \leq \mathbf{1} \end{cases} \tag{3.22}$$

also has a non-trivial solution for some (and therefore, for all) $\lambda > 0$, where $\tilde{D} = \{\tilde{d}_i;\ i \geq 0\}$ and

$$\tilde{d}_i = \tilde{q}_i - \sum_{j\neq i} \tilde{q}_{ij} \quad (\forall i \in E) \tag{3.23}$$

is the non-conservative quantity of the q-matrix $\tilde{Q}$. Moreover, we have

$$\lim_{i\to\infty} \lambda \sum_{j=0}^{\infty} \tilde{\phi}_{ij}(\lambda) = 0 \quad (\forall \lambda > 0), \tag{3.24}$$

where $\Phi(\lambda) = \{\tilde{\phi}_{ij}(\lambda);\ i, j \in E\}$ is the resolvent (i.e., the LaPlace transform) of the minimal $\tilde{Q}$-function $\tilde{F}(t) = \{\tilde{f}_{ij}(t);\ i, j \in E,\ t \geq 0\}$. In particular, the minimal $\tilde{Q}$-function is the Feller–Reuter–Riley transition function.

Proof. Let $\{v_i;\, i\in E\}$ be a non-trivial solution of Eq. (3.21) for a fixed $\lambda > 0$. Then

$$\begin{cases} \lambda v_i &= \sum\limits_{k=0}^{\infty} v_k q_{ki} \quad (\forall i \in E) \\ v_i &\geq 0 \quad (\forall i \in E) \\ 0 &< \sum\limits_{i=0}^{\infty} v_i < +\infty. \end{cases} \tag{3.25}$$

Define

$$x_0 = 0, \quad x_1 = \frac{v_0}{c}, \quad x_k = \frac{1}{c}\sum_{l=0}^{k-1} v_l \quad (k \geq 1), \tag{3.26}$$

where $c = \sum_{i=0}^{\infty} v_i$, and thus, $0 < c < +\infty$.

By (3.25), we have

$$\lambda \sum_{i=0}^{l-1} v_i = \sum_{i=0}^{l-1}\left(\sum_{k=0}^{\infty} v_k\, q_{ki}\right) = \sum_{k=0}^{\infty} v_k \left(\sum_{i=0}^{l-1} q_{ki}\right). \tag{3.27}$$

By noting (3.9), (3.26), and the fact that Q is conservative, we obtain

$$\begin{aligned} \lambda \cdot c \cdot x_l &= \sum_{k=0}^{\infty} v_k \left(-\sum_{i=l}^{\infty} q_{ki}\right) = \sum_{k=0}^{\infty} v_k \left(-\sum_{j=0}^{k} \tilde{q}_{lj}\right) \\ &= \sum_{j=0}^{\infty} \tilde{q}_{lj}\left(-\sum_{k=j}^{\infty} v_k\right) = \sum_{j=0}^{\infty} \tilde{q}_{lj}\left(-c + \sum_{k=0}^{j-1} v_k\right) \\ &= c \cdot \tilde{d}_l + c \cdot \sum_{j=0}^{\infty} \tilde{q}_{lj} x_j \quad (\forall l \in E), \end{aligned} \tag{3.28}$$

where $\tilde{d}_l = -\sum_{j=0}^{\infty} \tilde{q}_{lj} \quad (\forall l \in E)$ is the non-conservative quantity of $\tilde{Q}$.

Equation (3.28) shows that $\{x_k;\, k \geq 0\}$, defined in (3.26), is a solution of the equation

$$\begin{cases} \lambda x_k &= \tilde{d}_k + \sum\limits_{j=0}^{\infty} \tilde{q}_{kj}\, x_j \quad (\forall j \in E) \\ 0 &\leq x_k \leq 1 \quad (k \in E), \end{cases} \tag{3.29}$$

i.e., a solution of (3.22).

However, it is well known [2,10] that $(1 - \lambda\sum_{j=0}^{\infty} \tilde{\phi}_{ij}(\lambda);\; i \in E)$ is the maximal solution of Eq. (3.22), and hence,

$$0 \leq x_i \leq 1 - \lambda \sum_{j=0}^{\infty} \tilde{\phi}_{ij}(\lambda) \leq 1. \tag{3.30}$$

It is clear by (3.26) that $x_i \uparrow 1$ $(i \to \infty)$, and thus, by (3.30),

$$\lim_{i\to\infty} \lambda \sum_{j=0}^{\infty} \tilde{\phi}_{ij}(\lambda) = 0 \tag{3.31}$$

for the fixed $\lambda > 0$. It is then easy [10, 21] to have that (3.31) holds true for all $\lambda > 0$. It follows by (3.31) that the minimal $\tilde{Q}$-function is the Feller–Reuter–Riley function. □

Lemma 3.11 is a conclusion which is "independent" of other results. However, it holds a unique interest to us. We shall use Q^* to denote our q-matrix.

Lemma 3.12. *Let $Q^* = \{q^*_{ij};\, i, j \in E\}$ be a totally stable q-matrix (not necessarily conservative) defined on $E \times E$, where again $E = \{0, 1, 2, \dots\}$. Suppose Q^* satisfies the condition*

$$\sum_{k=0}^{j} q^*_{ik} \geq \sum_{k=0}^{j} q^*_{i+1,k} \quad (\forall j \neq i), \tag{3.32}$$

then the Feller minimal Q^*-function, $F^*(t) = \{f^*_{ij}(t);\, i, j \in E\}$ say, satisfies

$$\sum_{k=0}^{j} f^*_{ik}(t) \geq \sum_{k=0}^{j} f^*_{i+1,j}(t) \quad (\forall i, j \in E,\ \forall t \geq 0). \tag{3.33}$$

Remark 3.13. The proof of Lemma 3.12 is similar to that of Theorem 7.3.4 in [2]. Hence, we shall just mention the main steps and omit the details. For the differences, see Remark 3.14.

Proof. First, assume Q^* is uniformly bounded, i.e.,

$$\sup_{i \in E} q^*_i < +\infty. \tag{3.34}$$

Then, according to Proposition 2.2.10 in [2], we have an explicit formula for the minimal Q^*-function $F^*(t)$.

$$F^*(t) = e^{-\tau t} \cdot \sum_{n=0}^{\infty} \frac{(\tau t)^n}{n!} \cdot P^n, \tag{3.35}$$

where $\tau \geq \sup_{i \in E} q^*_i$ and

$$P = I + \frac{1}{\tau} Q^*. \tag{3.36}$$

Using condition (3.32), we can easily obtain

$$\sum_{k=0}^{j} p_{ik} \geq \sum_{k=0}^{j} p_{i+1,k} \quad (\forall i, j \in E), \tag{3.37}$$

where $P = \{p_{ij};\, i, j \in E\}$ is given in (3.36).

By induction, we can then obtain, for any $n \geq 0$,

$$\sum_{k=0}^{j} p^{(n)}_{ik} \geq \sum_{k=0}^{j} p^{(n)}_{i+1,k} \quad (\forall i, j \in E), \tag{3.38}$$

where $\{p^{(n)}_{ij};\, i, j \in E\} = P^n$.

Thus, by (3.35) and (3.38), we have that

$$\sum_{k=0}^{j} f^*_{ik}(t) \geq \sum_{k=0}^{j} f^*_{i+1,k}(t) \quad (\forall i, j \in E,\ \forall t \geq 0). \tag{3.39}$$

Now, we drop the bounded assumption (3.34) and define the truncated q-matrices ${}_NQ^* = \{{}_Nq^*_{ij};\, i, j\in E\}$ as follows:

$$
{}_Nq^*_{ij} = \begin{cases} q^*_{ij} & \text{if } i, j \in E_N \\ \sum\limits_{j=N}^{\infty} q^*_{ij} & \text{if } i \in E_N,\ j = N \\ 0 & \text{otherwise,} \end{cases}
$$

where $E_N = \{0, 1, 2, \ldots, N-1\}$.

It is easy to verify that ${}_NQ^*$ still satisfies (3.32) (for ${}_NQ$) and is uniformly bounded for each N. Thus, by using the result just proved, we have

$$
\sum_{k=0}^{j} {}_Nf^*_{ik}(t) \geq \sum_{k=0}^{j} {}_Nf^*_{i+1,k}(t) \quad (\forall i, j \in E, \forall t \geq 0). \tag{3.40}
$$

It is also easy to see that, by Proposition 2.2.14 in [2],

$$
{}_Nf^*_{ik}(t) \uparrow f^*_{ik}(t) \text{ as } N \to \infty \text{ for all } i, k, t \geq 0. \tag{3.41}
$$

Then, letting $N \to \infty$ in (3.40) yields (3.33). □

Remark 3.14. It should be pointed out that, in (3.41), the increasing property is so-called "essentially increasing", i.e., ${}_Nf^*_{ik}(t)$ increases for sufficient large N, and this N usually depends on (i, k). For (3.40), this will not cause problems since our sum is finite (for fixed j!). However, in Anderson's case, what he obtained is essentially

$$
\sum_{k=j}^{\infty} {}_Nf^*_{ik}(t) \leq \sum_{k=j}^{\infty} {}_Nf^*_{i+1,k}(t) \tag{3.42}
$$

rather than (3.40). Thus, monotone convergence theorem cannot simply be used in (3.42). Surely, if the minimal function is honest, then just reversing the direction of (3.42) could yield the correct conclusion. See Theorem 3.2 and Remark 3.3.

Now, we are ready to prove Theorem 3.7.

Proof. Suppose there exists a non-minimal monotone Q-function, $P(t)$ say. First, since Q is conservative, we see that $P(t)$ satisfies Kolmogorov backward equations. Then, by Corollary 2.6, Q must be monotone. Condition (i) follows.

Second, since $P(t)$ is a non-minimal monotone Q-function, by Theorem 3.1, we see that the minimal Q-function cannot be stochastically monotone. However, since we have just proved that the q-matrix Q is monotone, thus, by checking Theorem 3.2, we immediately obtain that Q cannot be regular which is equivalent to condition (ii).

Third, by Theorem 2.3, $P(t)$ satisfies Kolmogorov forward equations and we also know that $P(t)$ is the non-minimal stochastically monotone Q-function; condition (iii) must then hold true. Indeed, if (iii) does not hold true, then the only Q-function which satisfies Kolmogorov forward equations is Feller minimal, and thus by Theorem 2.3, our monotone Q-function would be Feller minimal, a contradiction. This ends the proof of the "only if" part.

Conversely, if conditions (i)–(iii) hold, then define the dual q-matrix $\tilde{Q}$ as in (3.6). We see that, by Lemma 3.9, $\tilde{Q}$ is a Feller q-matrix (see (3.7)) and satisfies (3.10), i.e.,

$$\sum_{k=0}^{j} \tilde{q}_{ik} \geq \sum_{k=0}^{j} \tilde{q}_{i+1,k} \quad (\forall j \neq i). \tag{3.43}$$

Then, by Lemma 3.12, we obtain that the minimal $\tilde{Q}$-function $\tilde{F}(t) = \{\tilde{f}_{ij}(t);\ i, j \in E, t \geq 0\}$ satisfies

$$\sum_{k=0}^{j} \tilde{f}_{ik} \geq \sum_{k=0}^{j} \tilde{f}_{i+1,k} \quad (\forall i, j \in E, \quad \forall t \geq 0). \tag{3.44}$$

Moreover, condition (iii) and Lemma 3.11 yield the fact that the minimal $\tilde{Q}$-function $\tilde{F}(t)$ is a Feller–Reuter–Riley function. This, together with (3.44), yields (see (1.7))

$$\tilde{F}(t) \in \tilde{D}, \tag{3.45}$$

which means that there exists a monotone transition function $P^*(t)$ say, such that $\tilde{F}(t)$ is the dual of $P^*(t)$. Let the q-matrix of $P^*(t)$ be $Q^* = \{q^*_{ij}\}$. Now, we claim that $Q^* = Q$. Indeed, since $\tilde{F}(t)$ is the dual of $P^*(t)$, we have

$$p^*_{ji}(t) = \sum_{k=0}^{j} \left(\tilde{f}_{ik}(t) - \tilde{f}_{i+1,k}(t)\right)$$

and thus,

$$q^*_{ji} = \sum_{k=0}^{j} \left(\tilde{q}_{ik} - \tilde{q}_{i+1,k}\right) \quad (\forall i, j \in E). \tag{3.46}$$

However, by the definition of $\tilde{Q}$ (see (3.6)),

$$\tilde{q}_{ji} = \sum_{k=i}^{\infty} \left(q_{ik} - q_{j-1,k}\right).$$

We have

$$\tilde{q}_{ik} - \tilde{q}_{i+1,k} = q_{ki} - q_{k-1,i}. \tag{3.47}$$

Substituting (3.47) into (3.46) yields

$$q^*_{ji} = q_{ji} \quad (\forall i, j \in E),$$

i.e., $Q^* = Q$. Thus, the monotone $P^*(t)$ is nothing but a Q-function which means $P^*(t) \in D(Q)$. However, $P^*(t)$ cannot be the Feller minimal Q-function, otherwise, by Theorem 3.2, Q would be regular, contradicting condition (ii). We have therefore proved that there exists a non-minimal monotone Q-function. □

Summing Theorems 3.1, 3.2, and 3.7, we obtain the following final result which answers satisfactorily the four basic questions posed in Sec 1.

Theorem 3.15 *For a given conservative q-matrix Q, there exists a stochastically monotone Q-function if and only if the following two conditions hold true simultaneously:*

(i) *Q is monotone;*
(ii) *either Q is or is not regular and the equation*

$$\begin{cases} V(\lambda I - Q) = \mathbf{0} \\ \mathbf{0} \leq V \in l \end{cases} \tag{3.48}$$

has a non-trivial solution for some (and therefore, for all) $\lambda > 0$. In all cases where there exists a stochastically monotone Q-function, it must be unique. Moreover, when Q is monotone and regular, this unique stochastically monotone Q-function is the Feller minimal Q-function, while if Q is monotone, not regular, and Eq. (3.48) has a non-trivial solution, then this unique stochastically monotone Q-function is non-minimal which can be constructed via the dual $\tilde{Q}$-function (the minimal $\tilde{Q}$-function) and formula (1.3). □

Remark 3.16. It is well known that the Feller minimal Q-function can be constructed asymptotically via Q directly for any q-matrix Q. It is important to note that, although the monotone q-function may not be Feller minimal, any monotone Q-function can be constructed via the Feller minimal $\tilde{Q}$-function, together with the simple formula (1.3) by Theorem 3.15. Hence, we also have answered Question 4 in Sec. 1.

4. Examples

In this section, we use two simple but important examples to illustrate our results: birth-death processes and branching processes.

Example 4.1. Birth-Death process. Recall that a conservative q-matrix Q of a birth-death process takes the following form:

$$q_{ij} = \begin{cases} b_i & \text{if } j = i+1, \ i \geq 0 \\ a_i & \text{if } j = i-1, \ i \geq 1 \\ -(a_i + b_i) & \text{if } j = i, \ i \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $a_i > 0, \ b_i > 0 \ (\forall i \geq 0)$, except that

$$a_0 = 0. \tag{4.2}$$

Further, recall that a further classification of birth-death processes, according to the boundary behavior is as follows: The boundary point is called

(i) *regular*, if both of the following equations (4.3) and (4.4), i.e.,

$$\begin{cases} (\lambda I - Q)U = \mathbf{0} \\ \mathbf{0} \leq U \leq \mathbf{1} \end{cases} \tag{4.3}$$

and

$$\begin{cases} V(\lambda I - Q) = \mathbf{0} \\ \mathbf{0} \leq V \in l \end{cases} \tag{4.4}$$

have non-trivial solutions;
(ii) *exit*, if only (4.3) has a non-trivial solution;
(iii) *entrance*, if only (4.4) has a non-trivial solution;
(iv) *natural*, if neither (4.3) nor (4.4) has a non-trivial solution.

Note that, for a birth-death q-matrix, we have convenient conditions to check whether (4.3) or (4.4) has a non-trivial solution. Indeed, (see [2] or [21]) Equation (4.3) has a non-trivial solution if and only if

$$\sum_{n=1}^{\infty} \frac{1}{a_n \cdot \pi_n} \sum_{m=0}^{n-1} \pi_m < +\infty,$$

while (4.4) has a non-trivial solution if and only if

$$\sum_{n=0}^{\infty} \frac{1}{b_n \cdot \pi_n} \sum_{m=n+1}^{\infty} \pi_m < +\infty,$$

where

$$\pi_n = \begin{cases} 1 & \text{if } n = 0 \\ \frac{b_0 b_1 \cdots b_{n-1}}{a_1 a_2 \cdots a_n} 1 & \text{if } n \geq 1 \end{cases}$$

are potential coefficients.

It is also easy to check that Q is always monotone and is a Feller q-matrix.

Note that, by the above classification, we know that a birth-death q-matrix is regular if and only if the boundary point is entrance or natural. Also, note that there exists easy checking conditions for classification of boundary points.

Considering all these facts, we now have

Theorem 4.1. *For a given birth-death q-matrix Q,*
(i) *if the boundary point is natural or entrance, then the Feller minimal Q-function is stochastically monotone and honest, and is the only stochastically monotone Q-function*;
(ii) *if the boundary point is regular, then the Feller minimal Q-function is not stochastically monotone. However, there exists another non-minimal stochastically monotone Q-function which is honest and is the only monotone Q-function*;
(iii) *if the boundary point is exit, then there exists no monotone Q-function.*

Proof. See the conclusions of Theorems 3.1, 3.2, and 3.7. □

Remark 4.2. Although there may not exist dual birth-death processes, the dual q-matrix $\tilde{Q}$ of any birth-death q-matrix definitely exists. An easy calculation shows that $\tilde{Q}$ is also a birth-death q-matrix but interchanging the "birth" and "death" rates. Indeed, for Q in (4.1), $\tilde{Q}$ takes the form of

$$\tilde{q}_{ij} = \begin{cases} a_i & \text{if } j = i+1 \\ b_{i-1} & \text{if } j = i-1 \\ -(a_i + b_{i-1}) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

We can further verify that there exists a "symmetric" relation between the boundary points of Q and $\tilde{Q}$. Actually, we have the following relations for the boundary points:

$$\tilde{Q} \text{ natural} \Leftrightarrow Q \text{ natural}; \ \tilde{Q} \text{ regular} \Leftrightarrow Q \text{ regular};$$

but

$$\tilde{Q} \text{ exit} \Leftrightarrow Q \text{ entrance}; \ \tilde{Q} \text{ entrance} \Leftrightarrow Q \text{ exit}.$$

Remark 4.3. Our birth-death example shows that our conclusions are not the same as those in [2]. First, we have shown that, for a birth-death q-matrix Q (even if conservative), the corresponding minimal Q-function may not be stochastically monotone and there may not even exist any stochastically monotone Q-functions, contradicting to Anderson's conclusion [2, p. 251]. Second, any birth-death q-matrix is a monotone Reuter q-matrix (a conservative Feller q-matrix must be Reuter) and there does exist a non-minimal stochastically monotone birth-death function, contradicting Corollary 4.3 in [2].

Remark 4.4. Note that, for a given Q in (4.1), the dual q-matrix $\tilde{Q}$ in (4.5) is itself monotone and conservative, and thus, we may consider the existence problem for the monotone $\tilde{Q}$-function. Using the language of dual process, we are referring to the "re-dual-process", i.e., the dual of a dual process. If such a "re-dual" process does exist, we shall call the original Q-process "redualable". In particular, if this re-dual process is just the original process (with possibly some one-to-one mappings between state spaces!), we shall call the original process "self-dualable". We will not have a formal discussion here (even for definitions) but just give the following conclusion informally.

Proposition 4.5. *For a given birth-death q-matrix Q in (4.1), there exists a "re-dualable" Q-process if and only if the boundary point of Q is natural, and in such cases, it is actually "self-dualable".*

Proof. It is simple and thus omitted. □

Example 4.2. Branching process example. Recall that a branching q-matrix Q is given by

$$q_{ij} = \begin{cases} i \cdot b_{j-i+1} & \text{if } j \geq i+1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

where $b_j \geq 0$ $(j \neq 1)$, $b_j \not\equiv 0$ $(j \geq 1)$, and

$$\sum_{j=0}^{\infty} b_j = 0. \tag{4.7}$$

Hence, any branching q-matrix is conservative. It is now easy to prove the following lemma.

Lemma 4.6. *Let Q be a q-matrix defined in (4.6)–(4.7). Then*

(i) *Q is monotone;* (4.8)

(ii)

$$\begin{cases} V(\lambda I - Q) = \mathbf{0} \\ \mathbf{0} \leq V \in l \end{cases} \tag{4.9}$$

has the only trivial solution for all $\lambda > 0$;

(iii) *Q is regular, i.e.,*

$$\begin{cases} (\lambda I - Q)U = \mathbf{0} \\ \mathbf{0} \le U \le \mathbf{1} \end{cases} \tag{4.10}$$

has the only trivial solution if and only if, for some (and therefore, for all) ε with $q < \varepsilon < 1$,

$$\int_\varepsilon^1 \frac{ds}{u(s)} = -\infty, \tag{4.11}$$

where $u(s) = \sum_{j=0}^\infty b_j s^j$ and q is the extinction probability of the branching process.

Note that, if $q = 1$, then Q is regular.

Proof. (i) By direct checking. Indeed, if $k \le i - 1$, then

$$\sum_{j=i-1}^\infty q_{ij} = \sum_{j=i-1}^\infty q_{i+1,j} = 0.$$

If $k = i$, then

$$\sum_{j=i}^\infty q_{ij} = -b_0 \le 0 = \sum_{j=i}^\infty q_{i+1,j}$$

while, if $k \ge i + 2$,

$$\sum_{j=k}^\infty q_{ij} = i \sum_{j=k}^\infty b_{j-i+1} \le (i+1) \sum_{j=k+1}^\infty b_{j-i} \le (i+1) \sum_{j=k}^\infty b_{j-i} = \sum_{j=k}^\infty q_{i+1,j}.$$

Hence, we have

$$\sum_{j=k}^\infty q_{ij} \le \sum_{j=k}^\infty q_{i+1,j} \quad (\forall k \ne i+1),$$

and thus, Q is monotone.

(ii) This is the uniqueness condition for branching processes. The proof can be seen in, for example, [9] by using uniqueness conditions for differential equations. A direct proof for (4.9) having no non-trivial solution can be seen in [4].

(iii) This is the honest condition for branching processes (see also [9]). □

Now, we have

Theorem 4.7. *For a branching q-matrix Q in (4.6)–(4.7), there exists a stochastically monotone Q-function if and only if Q is regular. Moreover, the only stochastically monotone Q-function is the Feller-minimal Q-function.*

Proof. If Q is regular, then, together with Lemma 4.6(i), we have that the minimal Q-function is stochastically monotone by Theorem 3.2.

Conversely, if there exists a stochastically monotone Q-function, then, by noting Lemma 4.6(ii) and Theorem 3.7, we see that this stochastically monotone Q-function cannot be non-minimal and thus must be Feller-minimal. Now, using Theorem 3.2 again, we have that Q is regular. The last part of Theorem 4.7 has also been proved. □

Note that, for a branching q-matrix in (4.6)–(4.7), it is actually downwardly skip-free. It is easy to check that the dual q-matrix of Q, $\tilde{Q}$ say, is upwardly skip-free. This fact is actually true for any monotone skip-free q-matrix.

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